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Blackbox computation of A_∞-algebras

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Abstract. Kadeishvili's proof of the minimality theorem [T. Kadeishvili, On the homology theory of fiber spaces, *Russ. Math. Surv.* **35**:3 (1980), 231–238] induces an algorithm for the inductive computation of an A_∞-algebra structure on the homology of a dg-algebra.

In this paper, we prove that for one class of dg-algebras, the resulting computation will generate a complete A_∞-algebra structure after a finite amount of computational work.

Keywords. A-infinity, strong homotopy associativity, inductive computation.

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To Tornike Kadeishvili

1 Introduction

A_∞-algebras have been in use in topology since their introduction by Stasheff [19]. Their applicability in an algebraic context was made clear by the Minimality theorem proved by Kadeishvili [11], then extended and reproved using the techniques of homological perturbation theory (see [10] for a good overview and [2] for one origin of the techniques) and explicit tree formulae [9, 18].

These structures are found in representation theory as well: using suitable A_∞-structures, Huebschmann has computed various group cohomology rings of groups given as extensions; the spectral sequence of the group extension under consideration is then an invariant of the A_∞-structure [3–6]. Furthermore both Keller [12–14] and Lu, Palmieri, Wu and Zhang [15, 16] have studied the use of A_∞-algebras for module categories and cohomology rings over specific algebras.

In many of the proofs of the Minimality theorem, a more or less explicitly stated version of a homological perturbation theory approached were used. This version uses decorated trees to give explicit formulas for the higher operations in terms of a deformation retract and the multiplication map from the original dg-algebra. The technique is present in the work of Huebschmann and Kadeishvili, and has very explicit treatments by Merkulov and by Lu, Palmieri, Wu and Zhang [9, 15, 16, 18].

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More details on the history of A_∞-algebras may be found in Huebschmann's survey talk [7] as well as in [8].

The techniques used to prove the Minimality theorem all yield explicit methods to compute an A_∞-algebra structure, and the main known techniques for producing such structures are rooted in available various proofs. All the methods rooted in homological perturbation theory yield complete structures, but at a price: they all require global information of a kind that in algebraic settings is not necessarily easy to come by. By contrast, the Kadeishvili algorithm, derived directly from Kadeishvili's original proof, only gives fragments of the A_∞-structure, but does so without the kind of global information expected in other methods. We merely need some sort of *blackbox*, capable of performing computations within the dg-algebra we start with to extract information about the resulting A_∞-algebra structure on its homology.

In this paper, we prove the following theorems, which together yield a strong reduction in the computational load in computing A_∞-algebra structures on particularly nice rings:

Theorem 1.1. *Suppose that A is a dg-algebra and suppose that*

A1. *there is an element $z \in H_*A$ generating a polynomial subalgebra and H_*A is free as a $\Bbbk[z]$-module and*

A2. *H_*A possesses a $\Bbbk[z]$-linear A_{n-1}-algebra structure induced by the dg-algebra structure on A, such that*

$$f_1(z)f_k(a_1,\dots,a_k) = f_k(a_1,\dots,a_k)f_1(z).$$

A3. *$b_1,\dots$ is a $\Bbbk[z]$-basis of H_*A, and that we have chosen $m_n(v_1,\dots,v_n)$ and $f_n(v_1,\dots,v_n)$ according to the Kadeishvili algorithm for all $v_i \in \{b_1,\dots\}$.*

Then a choice of $m_n(v_1,\dots,v_n)$ and $f_n(v_1,\dots,v_n)$ according to the Kadeishvili algorithm for all v_i taken from a $\Bbbk[z]$-basis for H_*A extends to a $\Bbbk[z]$-linear A_n-algebra structure for H_*A induced by the dg-algebra structure on A.

Theorem 1.2. *Let A be a dg-algebra. Suppose that in the computation of an A_{2q-2}-structure on H_*A, we have shown that $f_k = 0$ and $m_k = 0$ for all $q \le k \le 2q-2$ for some q.*

Then this A_{2q-2}-algebra structure extends to an A_∞-algebra structure with $f_k = 0$ and $m_k = 0$ for all $k \ge q$.

In the case where H_*A is a finite $\Bbbk[z]$-module, this reduces the workload necessary in order to inductively compute an A_n-algebra structure on H_*A to a finite workload. Additionally, we adapt the conditions of Theorem 4.1 in Theorem 4.2 to recompute the A_∞-algebra structure on the Yoneda Ext-algebra over the truncated polynomial ring originally computed by Madsen [17].

The paper is organized in five sections. Section 1 is this introduction. Section 2 defines A_∞-algebras and the properties used. Section 3 introduces the Minimality theorem, Kadeishvili's proof thereof and the algorithm inherent in the proof. In Section 4, the theorems discussed above are stated and proved, and finally in Section 5, we use the Computer Algebra System MAGMA to implement the techniques we have introduced.

2 A-infinity algebras

Let $\mathbb{k}$ be a field. An A_∞-algebra is a graded $\mathbb{k}$-vector space A equipped with a family $\{m_n\}_{n=1}^\infty$ of multilinear maps $m_n \in \operatorname{Hom}(A^{\otimes n}, A)$ of degree $2-n$ that satisfy the Stasheff family of identities

$$\mathrm{St}_n : \sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t}(\mathbb{1}^r \otimes m_s \otimes \mathbb{1}^t) = 0.$$

A morphism $f : A \to B$ of A_∞-algebras is a family $\{f_n\}_{n=1}^\infty$ of multilinear maps $f_n \in \operatorname{Hom}(A^{\otimes n}, A)$ of degree $1-n$ that satisfy the Stasheff family of morphism identities

$$\mathrm{St}_n^m : \sum_{r+s+t=n} (-1)^{r+st} f_{r+1+t}(\mathbb{1}^r \otimes m_s \otimes \mathbb{1}^t) = \sum (-1)^w m_q(f_{i_1} \otimes \cdots \otimes f_{i_q}),$$

where $i_1 + \cdots + i_q = n$ and $w = (q-1)(i_1-1) + \cdots + 2(i_{q-2}-1) + (i_{q-1}-1)$.

We also define an A_n-algebra to be a graded vector space with a family $\{m_k\}_{k=1}^n$ of multilinear maps satisfying the Stasheff identities $\mathrm{St}_1, \dots, \mathrm{St}_n$. A morphism of A_n-algebras is a family $\{f_k\}_{k=1}^n$ of multilinear maps satisfying the Stasheff identities $\mathrm{St}_1^m, \dots, \mathrm{St}_n^m$.

An A_n-, or A_∞-algebra is a dg-algebra if $m_n = 0$ for all $n \geq 3$.

For a dg-subalgebra $R \subseteq A$ of an A_n-, or A_∞-algebra, we call A an R-linear A_n-, or A_∞-algebra if $m_k \in \operatorname{Hom}_R(A^{\otimes k}, A)$ for all k. A morphism $f_* : A \to B$ of A_n-, or A_∞-algebras is called R-linear if $f_1(R)$ is a dg-subalgebra of B and for any $r \in R$ and $a_1, \dots, a_k \in A$,

$$f_k(a_1, \dots, ra_i, \dots, a_k) = f_1(r) f_k(a_1, \dots, a_i, \dots, a_k)$$

or, in other words, viewing B as an R-module through the map f_1, that $f_k \in \operatorname{Hom}_R(A^{\otimes k}, B)$ for all k.

3 Kadeishvili's algorithm

At the core of our approach to the computation of A_∞-algebras is the minimality theorem:

Theorem 3.1 (Minimality). *Let A be a dg-algebra so that H_*A is a free $\Bbbk$ module. There is an A_∞-algebra structure on H_*A and an A_∞-algebra quasi-isomorphism $f : H_*A \to A$ such that f_1 is a cycle-choosing quasi-isomorphism of dg-vector spaces, $m_1 = 0$ and m_2 is induced by the multiplication in A.*

The resulting structure is unique up to isomorphism.

If A has a unit 1, then the structure and quasi-isomorphism can be chosen to be strictly unital, i.e. such that $\forall k, \lambda \in \Bbbk, a_i \in A$

$$f_k(a_1,\dots,a_{i-1},\lambda\cdot 1,a_{i+1},a_k) = 0, \quad m_k(a_1,\dots,a_{i-1},\lambda\cdot 1,a_{i+1},a_k) = 0.$$

Proof. The following proof is from Kadeishvili's original paper [11]. We review most of it here as a basis for the algorithm.

Since A is a dg-algebra, there are only two operations in the A_∞-structure on A: m_1 and m_2. We shall denote m_1 by d and m_2 by the infix operator $\cdot$ or by juxtaposition.

To initialize an inductive definition of an A_∞-structure on H_*A, we pick $m_1 = 0$ and m_2 the induced multiplication on the coclasses. Furthermore, we let f_1 be some cycle-choosing $\Bbbk$-module homomorphism $H_*A \to A$.

Set $\Psi_2(a_1,a_2) = f_1(a_1a_2) - f_1(a_1)f_1(a_2)$. This is a boundary, since $f_1(a_1a_2)$ is defined to be a representative cycle of the homology class containing $f_1(a_1)f_1(a_2)$. Hence, there is some w such that $dw = \Psi_2(a_1,a_2)$. We define $f_2(a_1,a_2) = w$.

Now, for $n > 2$, write

$$\Psi_n(a_1,\dots,a_n) = \sum_{s=1}^{n-1}(-1)^{\varepsilon_1(a_1,\dots,a_n,s)} f_s(a_1,\dots,a_s)\cdot f_{n-s}(a_{s+1},\dots,a_n)$$
$$+\sum_{j=2}^{n-1}\sum_{k=0}^{n-j}(-1)^{\varepsilon_2(a_1,\dots,a_n,k,j)} f_{n-j+1}(a_1,\dots,a_k,m_j(a_{k+1},\dots,a_{k+j}),\dots,a_n),$$

where the expressions $\varepsilon_1(a_1,\dots,a_n,s) = s + (n-s+1)(|a_1|+\dots+|a_s|)$ and $\varepsilon_2(a_1,\dots,a_n,k,j) = k + j(n-k-j+|a_1|+\dots+|a_k|)$ are the signs in the Stasheff morphism axiom St_n^m with the Koszul signs introduced.

This Ψ_n is the complete expression of the Stasheff morphism axiom St_n^m, but with the two terms $f_1 m_n$ and $m_1 f_n$ removed. The central point of this proof is to fill in these terms.

By some tedious technical checking, we can confirm that the element $\Psi_n(a_1,\dots,a_n) \in \ker d$. Hence, $\Psi_n(a_1,\dots,a_n)$ belongs to some coclass $z \in H_*A$. We define

$$m_n(a_1,\dots,a_n) = z.$$

Sincenow $f_1(m_n(a_1,\dots,a_n))$ and $\Psi_n(a_1,\dots,a_n)$ are in the same coclass, there is some coboundary dw, $w \in A$, such that

$$f_1(m_n(a_1,\dots,a_n)) - \Psi_n(a_1,\dots,a_n) = dw.$$

We set

$$f_n(a_1,\dots,a_n) = w.$$

Since we defined everything precisely in order to match the Stasheff axioms, we obtain a structure that satisfies the Stasheff axioms.

For example, we note that

$$\begin{aligned}
\Psi_3(a,b,c) &= (-1)^{\varepsilon_1(a,b,c,1)} f_1(a) f_2(b,c) + (-1)^{\varepsilon_1(a,b,c,2)} f_2(a,b) f_1(c) \\
&\quad + (-1)^{\varepsilon_2(a,b,c,0,2)} f_2(m_2(a,b),c) \\
&\quad + (-1)^{\varepsilon_2(a,b,c,1,2)} f_2(a,m_2(b,c)) \\
&= (-1)^{1+1\cdot|a|} f_1(a) f_2(b,c) + (-1)^{2+2\cdot(|a|+|b|)} f_2(a,b) f_1(c) \\
&\quad + (-1)^{0+2\cdot(\cdots)} f_2(m_2(a,b),c) + (-1)^{1+2\cdot(\cdots)} f_2(a,m_2(b,c)) \\
&= -(-1)^{|a|} f_1(a) f_2(b,c) + f_2(a,b) f_1(c) \\
&\quad + f_2(m_2(a,b),c) - f_2(a,m_2(b,c)).
\end{aligned}$$

With regards to unitality, first we consider $m_2(1,a) = m_2(a,1) = a$; then $\Psi_2(1,a) = a - a = 0$ and $\Psi_2(a,1) = a - a = 0$. Thus we can safely choose $f_2(1,a) = f_2(a,1) = 0$.

Now consider Ψ_3. We have three cases to consider:

$$\begin{aligned}
\Psi_3(1,a,b) &= -(-1)^{|1|} f_1(1) f_2(a,b) + f_2(1,a) f_1(b) \\
&\quad + f_2(m_2(1,a),b) - f_2(1,m_2(a,b)) \\
&= -f_2(a,b) + 0 + f_2(a,b) - 0 = 0,
\end{aligned}$$

$$\begin{aligned}
\Psi_3(a,1,b) &= -(-1)^{|a|} f_1(a) f_2(1,b) + f_2(a,1) f_1(b) \\
&\quad + f_2(m_2(a,1),b) - f_2(a,m_2(1,b)) \\
&= -0 + 0 + f_2(a,b) - f_2(a,b) = 0, \\
\Psi_3(a,b,1) &= -(-1)^{|a|} f_1(a) f_2(b,1) + f_2(a,b) f_1(1) \\
&\quad + f_2(m_2(a,b),1) - f_2(a,m_2(b,1)) \\
&= -0 + f_2(a,b) + 0 - f_2(a,b) = 0.
\end{aligned}$$

Hence $\Psi_3 = 0$ whenever one input is a unit, thus $m_3 = 0$ when one input is a unit and we can choose $f_3 = 0$ when one input is a unit.

Consider now some $n > 3$. In the expression for Ψ_n, we have terms of the forms

$$f_i(a_1,\dots,a_i) \cdot f_j(a_{i+1},\dots,a_n)$$

and

$$f_i(a_1,\dots,m_k(a_j,\dots,a_{j+k}),\dots,a_n).$$

In the case that $a_1 = 1$ or $a_n = 1$, the 1 occurs inside some f_k or m_k, with $k > 1$, for all cases except for the terms $(-1)^{1+(n-2)|1|} f_1(1) f_{n-1}(a_2,\dots,a_n)$ and $(-1)^{0+2\cdot(\cdots)} f_{n-1}(m_2(1,a_2),\dots,a_n)$. These have the opposite signs, and their sum vanishes. Otherwise, in terms of first kind, the unit must occur as an argument to one of two f_*s, which consequently vanishes, and thus so does the entire term. Terms of second kind vanish whenever the unit occurs outside m_k. If the unit occurs within m_k, then we distinguish between $k > 2$ and $k = 2$. If $k > 2$, then m_k vanishes, by assumption. Thus we only need to consider the case $k = 2$. In this case two non-vanishing terms occur, namely

$$\begin{aligned}
&(-1)^j f_i(a_1,\dots,a_{i-1},m_2(1,a_{i+1}),\dots,a_n) \\
&\qquad\qquad = (-1)^j f_i(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n), \\
&(-1)^{j-1} f_i(a_1,\dots,m_2(a_{i-1},1),a_{i+1},\dots,a_n) \\
&\qquad\qquad = -(-1)^j f_i(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n).
\end{aligned}$$

Hence these terms cancel each other and we conclude that $\Psi_n = 0$. Thus $m_n = 0$ follows and we can safely choose $f_n = 0$.

Since the algorithm we derive does not use the internal structure of the proof of uniqueness of the operations, we refer the reader to [11] for a proof. □

This translates immediately into an algorithm for pointwise computation of A_∞-structure maps. For the computation of an A_∞-structure on H_*A, we need to fix some data for the entire computation. Central for this is the choice of a cycle-choosing map $f_1\colon H_*A \to A$. We need the map to send the classes to the cycles representing the classes, but any such choice will work. This is not the only choice made in the algorithm – for each computed $f_n(a_1,\dots,a_n)$, a choice is made of an appropriate coboundary dw, to make f_n a $\Bbbk$-module homomorphism. However, this is easily seen to be reliable, as H_*A is free over $\Bbbk$.

Algorithm 3.2 (Kadeishvili Algorithm). The algorithm takes as input a list of elements $a_1,\dots,a_n$ in H_*A, a cycle-choosing homomorphism $f_1\colon H_*A \to A$ as described above, and returns $m_n(a_1,\dots,a_n)$ and $f_n(a_1,\dots,a_n)$ fulfilling the Stasheff axioms St_n and St_n^m.

(1) If $n = 1$, return $m_1(a_1) = 0$ and $f_1(a_1)$ immediately.

(2) If $n = 2$, set $\Psi_2(a_1,a_2) = f_1(a_1)f_1(a_2)$ and $m_2(a_1,a_2) = a_1a_2$ and go to step 4. Otherwise, compute

$$\begin{aligned}
&\Psi_n(a_1,\dots,a_n)\\
&= \sum_{s=1}^{n-1}(-1)^{\varepsilon_1(a_1,\dots,a_n,s)}f_s(a_1,\dots,a_s)\cdot f_{n-s}(a_{s+1},\dots,a_n)\\
&+\sum_{j=2}^{n-1}\sum_{k=0}^{n-j}(-1)^{\varepsilon_2(a_1,\dots,a_n,k,j)}f_{n-j+1}(a_1,\dots,a_k,m_j(a_{k+1},\dots,a_{k+j}),\dots,a_n),
\end{aligned}$$

where the expressions $\varepsilon_1(a_1,\dots,a_n,s) = s + (n-s+1)(|a_1|+\dots+|a_s|)$ and $\varepsilon_2(a_1,\dots,a_n,k,j) = k + j(n-k-j+|a_1|+\dots+|a_k|)$ are the signs in the Stasheff morphism axiom St_n^m with the Koszul signs introduced.

Note that the values of f_k and m_k for $k < n$ may be computed recursively using subsequent calls to this algorithm. The recursion bottoms out since m_1, m_2 and f_1 are already given.

(3) By the proof of the Minimality theorem, the element $\Psi_n(a_1,\dots,a_n) \in A$ is a cycle. Hence, it belongs to some homology class x. Set $m_n(a_1,\dots,a_n) = x$.

(4) Since $m_n(a_1,\dots,a_n)$ is the homology class containing $\Psi_n(a_1,\dots,a_n)$, the cycle $f_1(m_n(a_1,\dots,a_n))$ is homologous to the cycle $\Psi_n(a_1,\dots,a_n)$. Thus $\Psi_n(a_1,\dots,a_n) - f_1(m_n(a_1,\dots,a_n))$ is a boundary, and we can pick an element y such that $dy = \Psi_n(a_1,\dots,a_n) - f_1(m_n(a_1,\dots,a_n))$. We set $f_n(a_1,\dots,a_n) = y$, and return the higher multiplication $m_n(a_1,\dots,a_n)$ and the quasi-isomorphism component $f_n(a_1,\dots,a_n)$.

4 Computational reduction

At first glance, computing A_∞-structures by blackbox method seems impractical due to a high degree of recursion and the multiple infinities involved, i.e. there are infinitely many arities to compute, and even H_*A tends to be infinite dimensional in group cohomology, hence so is $(H_*A)^{\otimes n}$ for all relevant values of n.

In specific cases, however, we are able to reduce the complexity of these computations to a manageable size. For especially well-behaved cohomology rings, we are able to reduce the computation of a full A_∞-structure to a finite problem.

We say that the induced A_∞-algebra structures found by the Minimality theorem are R-linear if the structure maps are R-linear, and the quasi-isomorphism is an R-linear A_∞-morphism.

Theorem 4.1. *Suppose that A is a dg-algebra and suppose that*

A1. *there is an element $z \in H_*A$ generating a polynomial subalgebra and H_*A is free as a $\Bbbk[z]$-module and*

A2. *H_*A possesses a $\Bbbk[z]$-linear A_{n-1}-algebra structure induced by the dg-algebra structure on A, such that*

$$f_1(z)f_k(a_1,\ldots,a_k) = f_k(a_1,\ldots,a_k)f_1(z).$$

A3. *$b_1,\ldots$ is a $\Bbbk[z]$-basis of H_*A, and that we have chosen $m_n(v_1,\ldots,v_n)$ and $f_n(v_1,\ldots,v_n)$ according to the Kadeishvili algorithm for all $v_i \in \{b_1,\ldots\}$.*

Then a choice of $m_n(v_1,\ldots,v_n)$ and $f_n(v_1,\ldots,v_n)$ according to the Kadeishvili algorithm for all v_i taken from a $\Bbbk[z]$-basis for H_*A extends to a $\Bbbk[z]$-linear A_n-algebra structure for H_*A induced by the dg-algebra structure on A.

Proof. Set $\zeta = f_1(z)$. We need to consider

$$\begin{aligned}
&\Psi_n(a_1,\ldots,za_i,\ldots,a_n)\\
&\quad= \sum \pm f_j(a_1,\ldots,za_i,\ldots,a_j)f_{n-j}(a_{j+1},\ldots,a_n)\\
&\qquad+ \sum \pm f_j(a_1,\ldots,a_j)f_{n-j}(a_{j+1},\ldots,za_i,\ldots,a_n)\\
&\qquad+ \sum \pm f_{n-j+1}(a_1,\ldots,za_i,\ldots,m_j(a_{k+1},\ldots,a_{k+j}),\ldots,a_n)\\
&\qquad+ \sum \pm f_{n-j+1}(a_1,\ldots,m_j(a_{k+1},\ldots,za_i,\ldots,a_{k+j}),\ldots,a_n)\\
&\qquad+ \sum \pm f_{n-j+1}(a_1,\ldots,m_j(a_{k+1},\ldots,a_{k+j}),\ldots,za_i,\ldots,a_n).
\end{aligned}$$

In each summand of this expression, the term za_i occurs within either f_j or m_j of lower arity than n. Hence, by assumption, we can commute z out to ζ. Since, also, ζ commutes with all f_n of lower arity, we find that

$$\Psi_n(a_1, \ldots, za_i, \ldots, a_n) = \zeta \Psi_n(a_1, \ldots, a_n).$$

Hence $m_n(a_1, \ldots, za_i, \ldots, a_n) = zm_n(a_1, \ldots, a_n)$ follows. To finalize the argument, we need to find a boundary h of the element $\zeta(\Psi_n - f_1 m_n)(a_1, \ldots, a_n)$, given a boundary h' of $(\Psi_n - f_1 m_n)(a_1, \ldots, a_n)$. Note that

$$d(\zeta h') = (d\zeta)h' + \zeta dh' = \zeta(\Psi_n - f_1 m_n)(a_1, \ldots, a_n);$$

since $\zeta = f_1(z)$, f_1 chooses cycles and hence $d\zeta = 0$. Thus $\zeta h'$ is such a boundary. □

Theorem 4.2. *Suppose that R is a finite $\Bbbk$-algebra and that*

B1. *X is a periodic resolution of period π of finitely generated R-modules, and that $A = \mathrm{End}_R(X)$ is the endomorphism dg-algebra of X. Suppose further that there is some element $0 \neq z \in H_*A$ such that we can choose $f_1(z) = \zeta$, a periodic map of period π with each $\zeta_n = \mathrm{Id}$.*

B2. *for all $k < n$ we have constructed m_k and f_k such that* A2 *holds.*

B3. *$b_1, \ldots, b_t$ is a $\Bbbk[z]$-basis for H_*A and for all $v_1, \ldots, v_n$ chosen such that each $v_i \in \{b_1, \ldots, b_t\}$ we know that $f_n(v_1, \ldots, v_n)$ is periodic of period π for all choices $v_1, \ldots, v_n$.*

Then, from B1*, we can infer that z generates a polynomial subalgebra of H_*A and H_*A is free over $\Bbbk[z]$.*

From B1, B2 *and the fact that $\zeta f_k(a_1, \ldots, a_k) = f_k(a_1, \ldots, a_k)\zeta$, the homotopies $f_n(a_1, \ldots, a_k)$ are periodic of period π.*

In addition, B3 *implies that for all $a_1, \ldots, a_n \in H_*A$, the map $f_n(a_1, \ldots, a_n)$ is periodic of period π and*

$$\begin{aligned} m_n(a_1, \ldots, za_i, \ldots, a_n) &= zm_n(a_1, \ldots, a_n), \\ f_n(a_1, \ldots, za_i, \ldots, a_n) &= \zeta f_n(a_1, \ldots, a_n), \\ f_n(a_1, \ldots, a_n)\zeta &= \zeta f_n(a_1, \ldots, a_n). \end{aligned}$$

Proof. If some ζ^N is null-homotopic, we can use the periodicity of X to lower the degree of the null-homotopy, thereby inducing a null-homotopy for ζ. However, we have assumed that $z \neq 0$. Hence $f_1(z) = \zeta$ is not null-homotopic. Thus $\Bbbk[z]$ is a polynomial subalgebra of H_*A. We set $I = H^{\geq 1}A$. This is an ideal in H_*A, and we can find the $\Bbbk$-vector space $J = I/I^2$ of indecomposables. We can pick a basis $b_1, \dots, b_r$ of $J/(z)$. Every b_i has a representative in J, hence a representative that is not divisible by z. Furthermore, $b_1, \dots, b_r, z$ generate H_*A as a $\Bbbk$-algebra. Suppose now that we had some dependency $\sum_i a_i b_i = 0$ over $\Bbbk[z]$. Then $z|a_i$ for all a_i, since otherwise b_i would not form a basis of $J/(z)$. But then we could divide the dependency by an appropriate power of z and get a dependency involving the indecomposables. Hence H_*A is free over $\Bbbk[z]$.

From the condition $f_k(a_1, \dots, a_k)\zeta = \zeta f_k(a_1, \dots, a_k)$ we get by setting $d = |f_k(a_1, \dots, a_k)|$, that $(f_k(a_1, \dots, a_k)\zeta)_n = f_k(a_1, \dots, a_k)_n \zeta_{n+d}$ and that $(\zeta f_k(a_1, \dots, a_k))_n = \zeta_n f_k(a_1, \dots, a_k)_{n+\pi}$. The equality of chain maps forces the equality $f_k(a_1, \dots, a_k)_n \zeta_{n+d} = \zeta_n f_k(a_1, \dots, a_k)_{n+\pi}$ and, by the definition of ζ, we are left with $f_k(a_1, \dots, a_k)_n = f_k(a_1, \dots, a_k)_{n+\pi}$.

Since $b_1, \dots, b_t$ form a $\Bbbk[z]$-linear basis of H_*A, any element $a \in H_*A$ has a unique decomposition into a $\Bbbk[z]$-linear combination of b_i.

By Theorem 4.1, all the commutativity relations hold.

For the periodicity of $f_n(a_1, \dots, a_n)$, consider the terms of the difference $\Psi_n(a_1, \dots, a_n) - f_1 m_n(a_1, \dots, a_n)$. Each term in this expression is either a composition of periodic maps of period π, or a periodic map of period π, by the assumptions on all f_k. Hence $\Psi_n(a_1, \dots, a_n) - f_1 m_n(a_1, \dots, a_n)$ is periodic of period π.

Finally, by assumption BB3, $f_n(v_1, \dots, v_n)$ is periodic of period π, for all choices of $v_1, \dots, v_n \in \{b_1, \dots, b_t\}$. Hence $f_n(v_1, \dots, zv_k, \dots, v_n) = \zeta f_n(v_1, \dots, v_n)$, and by Theorem 4.1, the resulting homotopy $f_n(a_1, \dots, a_n)$ is given by composing some ζ^s, which has period π, with $f_n(v_1, \dots, v_n)$, which is also periodic of period π. □

Theorem 4.3. *Let A be a dg-algebra. Suppose that in the computation of an A_{2q-2}-structure on H_*A, we have shown that $f_k = 0$ and $m_k = 0$ for all $q \le k \le 2q-2$ for some q.*

Then this A_{2q-2}-algebra structure extends to an A_∞-algebra structure with $f_k = 0$ and $m_k = 0$ for all $k \ge q$.

Proof. The proof follows by induction. Suppose that $\kappa > 2q-2$, and that we have already proved $f_k = 0$ and $m_k = 0$ for all $q \le k < \kappa$. In the computational step where we compute f_κ and m_κ, we start by considering Ψ_κ. This expression has two kinds of terms.

First, there are terms of the form $f_i \cdot f_{\kappa-i}$. Since $\kappa > 2q-2$, either $i \ge q$ or $\kappa - i \ge q$. Hence, by the induction hypothesis, $f_i \cdot f_{\kappa-i} = 0$. Second, there are terms of the form $f_i \circ_j m_{\kappa-i+1}$. Again, either $\kappa - i + 1 \ge \kappa - i \ge q$ or $i \ge q$. Hence, by hypothesis, either $f_i = 0$ or $m_{\kappa-i+1} = 0$. Hence $\Psi_\kappa = 0$. Thus we can choose $m_\kappa = 0$ and $f_\kappa = 0$. This proves the induction step and completes the proof. $\square$

4.1 Application: Cohomology of cyclic groups

Suppose $\Bbbk$ is a finite field and $R = \Bbbk[\alpha]/(\alpha^q)$. Then $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ is, for a power q of the characteristic p of $\Bbbk$, the ring $H^*(C_q, \Bbbk)$. For $q \geq 4$, it has a finite presentation, as a ring, given by $\Lambda^*(x) \otimes \Bbbk[y]$. We can choose a minimal periodic free resolution (P_*, d) of $\Bbbk$ using R-modules on the form

$$\begin{tikzcd}
\cdots \arrow[r] & R \arrow[r, "\cdot\alpha"] & R \arrow[r, "\cdot\alpha^{q-1}"] & R \arrow[r, "\cdot\alpha"] & R \arrow[r, "\cdot\alpha^{q-1}"] & R \arrow[r, "\cdot\alpha"] & R \arrow[r] & R \arrow[r] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

Now, $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ is the homology of the dg-algebra of graded module maps $P_* \to P_*$, with the induced differential $\partial f = df - (-1)^{|f|} fd$, and thus we can find representatives for the classes x and y given by ξ and η:

$$\begin{tikzcd}
\ldots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot\alpha^{q-2}"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot\alpha^{q-2}"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] \arrow[dr, "\epsilon"] & \Bbbk G \arrow[r] & \Bbbk \arrow[r] & 0 \\
\ldots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\epsilon"] & \Bbbk \arrow[rr] & & 0
\end{tikzcd}$$

$\xi :$.

$$\begin{tikzcd}
\ldots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot 1"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 1"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot 1"] \arrow[dr, "\epsilon"] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r] & \Bbbk \arrow[r] & 0 \\
\ldots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\epsilon"] & \Bbbk \arrow[r] & 0
\end{tikzcd}$$

$\eta :$.

The map η fulfills all the requirements for ζ in Theorem 4.2, and hence, as long as all computed homotopies have periodicity dividing 2, it will suffice to consider parameter sets taken from $\{1, x\}$ when computing higher multiplications. Furthermore, since we may compute a strictly unital A_∞-algebra structure, the only parameter sets we need to consider are of the form $x \otimes \cdots \otimes x$. By $\Bbbk[y]$-linearity, all other values follow. The A_∞-algebra structure on this cohomology ring computed by Dag Madsen [17] agrees with the A_∞-algebra structure found by the techniques in this paper.

Performing the Kadeishvili algorithm in this setting yields the maps $f_m(x, \ldots, x)$ as given by h in

$$\begin{tikzcd}
\ldots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-\alpha^{q-1-m})"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-\alpha^{q-1-m})"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] \arrow[dr, "\epsilon"] & \Bbbk G \\
\ldots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\epsilon"] & \Bbbk
\end{tikzcd}$$

$h :$.

and correspondingly $m_m(x,\dots,x) = 0$ for all $m < q-1$. Then $f_{q-1}(x,\dots,x)$ is given by h in

$$h : \quad \begin{tikzcd}
\dots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-1)"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] \arrow[d, "\cdot(-1)"] & \Bbbk G \arrow[r, "\cdot\alpha"] \arrow[d, "\cdot 0"] \arrow[dr, "\epsilon"] & \Bbbk G \\
\dots \arrow[r] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\cdot\alpha^{q-1}"] & \Bbbk G \arrow[r, "\cdot\alpha"] & \Bbbk G \arrow[r, "\epsilon"] & \Bbbk
\end{tikzcd} .$$

and thus $m_q(x,\dots,x) = y$ and $f_q(x,\dots,x) = 0$. Computing further will reveal $f_m(x,\dots,x) = 0$ and $m_m(x,\dots,x) = 0$ for all $q+1 \le m \le 2q$, which completes the computation of an A_∞-algebra structure on $\mathrm{Ext}^*_R(\Bbbk,\Bbbk)$.

5 Implementation

This approach is implemented in the MAGMA computer algebra system [1] as a component in the computational group cohomology modules. The implementation of the Kadeishvili algorithm is expected to work with some $\mathrm{End}_R(X)$, but the supporting homological algebra functionality was built to compute group cohomology rings. Hence, currently, the module will only work smoothly computing A_∞-algebra structures on cohomology rings of finite groups.

The user interface has three functions at its core:

`AInfinityRecord(G,n)` Yields a computation object storing internal information for the computations. Expects a finite group G and the length of a partial projective resolution of the trivial module $\Bbbk$ over the group algebra $\Bbbk G$.

`HighProduct(Aoo,lst)` Computes $m_k(a_1,\dots,a_k)$. Expects a computation object `Aoo`, as produced by `AInfinityRecord` and a sequence `lst` of length k of elements of the cohomology ring stored as `Aoo‘S`. Returns $m_k(a_1,\dots,a_k)$ as an element of `Aoo‘S`.

`HighMap(Aoo,lst)` Computes the quasi-isomorphism component $f_k(a_1,\dots,a_k)$. Expects a computation object `Aoo`, as produced by `AInfinityRecord` and a sequence `lst` of length k of elements of `Aoo‘S`. Returns $f_k(a_1,\dots,a_k)$ as a chain map endomorphism of the partial projective resolution stored in `Aoo‘P`.

Bibliography

[1] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *Computational algebra and number theory*, London, 1993. *J. Symbolic Comput.* **24**:3–4 (1997), 235–265.

[2] J. Huebschmann, The homotopy type of $F\Psi^q$. The complex and symplectic cases, in: *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory*, Part I, II (Boulder, CO, 1983), Contemp. Math. 55, 487–518, Am. Math. Soc., Providence, RI, 1986.

[3] J. Huebschmann, Cohomology of nilpotent groups of class 2, *J. Algebra* **126**:2 (1989), 400–450.

[4] J. Huebschmann, The mod-p cohomology rings of metacyclic groups, *J. Pure Appl. Algebra* **60**:1 (1989), 53–103.

[5] J. Huebschmann, Perturbation theory and free resolutions for nilpotent groups of class 2, *J. Algebra* **126**:2 (1989), 348–399.

[6] J. Huebschmann, Cohomology of metacyclic groups, *Trans. Am. Math. Soc.* **328**:1 (1991), 1–72.

[7] J. Huebschmann, Origins and breadth of the theory of higher homotopies, Festschrift in honor of M. Gerstenhaber's 80-th and Jim Stasheff's 70-th birthday, to appear in *Prog. Math.*, `arXiv:0710.2645` [math.AT].

[8] J. Huebschmann, On the construction of A_∞-structures, *Georgian Math. J.* **17**:1 (2010), 161–202, `arXiv:0809.4791`.

[9] J. Huebschmann and T. Kadeishvili, Small models for chain algebras, *Math. Z.* **207**:2 (1991), 245–280.

[10] L. Johansson and L. Lambe, Transferring algebra structures up to homology equivalence, *Math. Scand.* **89**:2 (2001), 181–200.

[11] T. V. Kadeishvili, On the theory of homology of fiber spaces *Uspekhi Mat. Nauk* **35**:3(213) (1980), 183–188 (Russian); translation in *Russ. Math. Surv.* **35**:3 (1980), 231–238, `arXiv:math/0504437`.

[12] B. Keller, Introduction to A-infinity algebras and modules, *Homol. Homotopy Appl.* **3**:1 (2001), 1–35 (electronic).

[13] B. Keller, Addendum to: Introduction to A-infinity algebras and modules [*Homol. Homotopy Appl.* **3**:1 (2001), 1–35, MR1854636 (2004a:18008a)], *Homol. Homotopy Appl.* **4**:1 (2002), 25–28 (electronic).

[14] B. Keller, A-infinity algebras in representation theory, *Representations of Algebra,* Vol. I, II, 74–86, Beijing Norm. Univ. Press, Beijing, 2002.

[15] D. M. Lu, J. H. Palmieri, Q. S. Wu and J. J. Zhang, A_∞-algebras for ring theorists, *Proceedings of the International Conference on Algebra*, *Algebra Colloq.* **11**:1 (2004), 91–128.

[16] D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, A-infinity structure on Ext-algebras, *J. Pure Appl. Algebra* **213**:11 (2009), 2017–2037, `arXiv:math/0606144`.

[17] D. Madsen, *Homological Aspects in Representation Theory*, Ph.D. thesis, Norges Teknisk-Naturvitenskapelige Universitet, 2002.

[18] S. A. Merkulov, Strong homotopy algebras of a Kahler manifold, *Int. Math. Res. Not.* **1999**:3 (1999), 153–164.

[19] J. D. Stasheff, Homotopy associativity of H-spaces, I, II, *Trans. Am. Math. Soc.* **108** (1963), 275–292, ibid. **108** (1963), 293–312.

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