Formalising a Proof of Contraction Admissibility for G4ip

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1 Introduction

We will, in this document, give a report of a formalisation, in *Isabelle*, of Contraction admissibility for the calculus **G4ip**. Specifically, we formalise sections of [3], which goes on to directly prove Cut admissibility as well. We do not prove Cut admissibility directly.

2 G4ip

G4ip is similar to **G3ip**, except where implication is concerned. The left rule for implication of **G3ip**, for instance given in [5], is

$$\frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C}\ L\supset$$

whereas in **G4ip**, independently discovered[1] by Vorob'ev [6], Hudelmaier [4] and Dyckhoff[2], we have four separate rules depending on the structure of the antecedent of the implication (where P is an atomic formula):

$$\frac{\Gamma, P, B \Rightarrow E}{\Gamma, P, P \supset B \Rightarrow E}\ L\supset 0$$

$$\frac{\Gamma, C \supset (D \supset B) \Rightarrow E}{\Gamma, (C \wedge D) \supset B \Rightarrow E}\ L\supset\wedge$$

$$\frac{\Gamma, C \supset B, D \supset B \Rightarrow E}{\Gamma, (C \vee D) \supset B \Rightarrow E}\ L\supset\vee$$

$$\frac{\Gamma, C, D \supset B \Rightarrow D \quad \Gamma, B \Rightarrow E}{\Gamma, (C \supset D) \supset B \Rightarrow E}\ L\supset\supset$$

[1] or invented, depending on your philosophical viewpoint

One of the main measures over which we perform induction is the *weight* of a formula. This is defined as follows:

$$\begin{aligned}
&w(\bot) = 0,\\
&w(P) = 1 \text{ for any atomic formula } P,\\
&w(A \supset B) = 1 + w(A) + w(B),\\
&w(A \wedge B) = 2 + w(A) + w(B),\\
&w(A \vee B) = 3 + w(A) + w(B).
\end{aligned}$$

Our development in Isabelle is adapted from [1]; each node in a derivation tree contains a parameter indicating its *height*. At several points, we perform an induction on the ordered pair $(w(A), n)$, where A is a formula, and n is the height at which that formula appears.

3 The Formal Proof

We prove several auxiliary lemmata before Contraction admissibility. Most of these are standard in such a formalisation, such as depth preserving weakening and various inversion results. As such, they are of little interest, and can be seen in the proof script[2]. Similarly, we also prove the generalised axiom ($\Gamma, A \Rightarrow A$ for *any* A) is derivable, and also *modus ponens* is derivable. The former is by induction on the weight of A. It should be noted here that [3] misses out a case of the proof. When A is an implication, we do further case analysis on the structure of the antecedent. So we have $A = B \supset C$, and the five cases

$$\begin{aligned}
&B = P \text{ for some propositional variable } P,\\
&B = \bot,\\
&B = G \wedge H,\\
&B = G \vee H,\\
&B = G \supset H
\end{aligned}$$

There is no case in the informal proof corresponding to $B = \bot$. However, it is a straightforward result, as the following shows:

$$\dfrac{\dfrac{}{\Gamma, \bot \supset C, \bot \Rightarrow C}\,L\bot}{\Gamma, \bot \supset C \Rightarrow \bot \supset C}\,R\supset$$

This is a minor omission, but the fact that it is an omission which was found through formalisation could be seen as justification in itself for the decision to formalise the paper.

[2] available from www.dcs.st-andrews.ac.uk/~pc

There are two more lemmata which must be proved before the main result can be shown, however working through them gives one no new insights into how *Isabelle* can be used.

3.1 Contraction Admissibility

The theorem which we are formalising the proof of is as follows: The rule

$$\frac{\Gamma, A, A \Rightarrow E}{\Gamma, A \Rightarrow E}\ \text{Contr}$$

is admissible in **G4ip**.

The proof in [3] is structured slightly differently from the one in the formalisation. In the informal proof, the structure is that of "first, do case analysis on A. Then, for each A, on the rule used to derive the premiss", because the induction is on the weight of the contracted formula A, and for each weight, on the height n of the derivation of the premiss. This approach is possible in the formalisation (indeed, there are two versions, one of which uses this structure), however the informal proof relies on techniques not readily available in *Isabelle* with which to reason. More specifically, the informal proof covers many of the cases with the line

> If $n > 0$ and A is not principal, apply the inductive hypothesis to the premises (sic.) and then use the rule again.

This is a standard, and perfectly allowable, technique, however it hides the fact that this is proving most of the cases, which need to be written out in a formalisation in full. To give concrete figures, the proof was first formalised using this structure (formula weight, then last rule used), and there were 73 cases, of which 64 were non-principal.

For the formalisation we then used a different method: "first, do case analysis on the last rule used. Then, for each rule, we have two cases: either A is principal, or it is non-principal." This may seem like not much of a difference, however it is easy to formalise when a formula is not principal for a rule, but not so easy to formalise when a rule is not principal for a formula. We are given the rule, and thus can perform the contraction in the premiss(es) easily; without the rule being given, this is difficult. It also has the side effect of reducing the number of cases; now there are only 22 (two for each rule), rather than the 74. As can easily be imagined, this reduces the length of the formalisation by quite a lot. The first method has a proof which is 629 lines long in *Isabelle*, the second method has a proof of only 239 lines.

4 Conclusions

There are two important facts to be taken away from this formalisation. One is that it is of the utmost importance to structure a proof correctly to reduce its length, and hence increase its readability. The second is that proving inversion lemmata is tedious. In an ideal world, when we define some rules in *Isabelle*, the system would prove the corresponding lemma for us (if, of course, it was possible). It is this second problem I will investigate first.

References

[1] P. Chapman. A formalised proof of cut admissibility. University of St Andrews Computer Science Research Report, available at www.dcs.st-andrews.ac.uk/∼pc, 2008.

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