Cut Elimination in G3i

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Notational Conventions

We follow the notation of, e.g. [DLK06], rather than that of [TS00].
So, the statement A *implies* B is written as $A \supset B$.

Why we want to Eliminate Cut

- The *Sub-Formula Property* holds in Cut-free systems.
  - No formulae disappear.
  - $\Rightarrow \bot$ is not derivable, so syntactically consistent.

- The fact that some formulae are *underivable* is an easy consequence of cut-free systems.
  - $\Rightarrow P \vee \neg P$ is underivable.

- Pierce's Law is underivable.

- The logical connectives for intuitionistic logic cannot be expressed in terms of the others. e.g. $\neg A \supset B \Rightarrow A \vee B$ is underivable.

• Derivability is decidable for **G3ip**.

Correspondence with Natural Deduction

We know that natural deduction introduction rules correspond to the right rules of the sequent calculus. Similarly, the elimination rules correspond to the left rules of the calculus.

Also, there are *detours* in natural deduction, where an introduction rule is immediately followed by the corresponding elimination rule. The corresponding detours in intuitionistic sequent calculus are the instances of the cut rule.

In this way, a cut-free proof in sequent calculus can be seen as a normal natural deduction proof.

Lambda Calculus, Sequent Calculus and Natural Deduction

The simply typed lambda calculus, $\lambda_{\rightarrow}$, is strongly normalising.

Normalising a proof in natural deduction can be seen as equivalent to a beta-reduction in $\lambda_{\rightarrow}$. Proving normalisation of a proof in natural deduction becomes almost trivial.

Due to the close correspondence between sequent calculus and natural deductions, a reformulation of the typing rules likewise shows that Cut free proofs are equivalent to normal forms in the $\lambda_{\rightarrow}$.

Lambda Calculus and the Sequent Calculus

This is not very satisfactory.

Sometimes, a deduction may use the cut rule even if the corresponding λ-term is in normal form; the beta reduction may delete terms which contain redexes. All we know is that there is a cut free proof; $\lambda_{\rightarrow}$ does not supply us with one.

To rectify this, *explicit substitution calculi* were developed, most notably $\lambda\mathtt{x}$. Here, substitutions are given new term constructors, and the cut rule is one of these substitutions.

Constructive Cut Elimination Proofs

What we really want is a process that allows us to build the cut elimination ourselves.

For each Cut formula, and the rule that derives the Cut formula, we need a systematic way to eliminate the instance of the Cut rule.

This could be by replacing it with “smaller” cuts in one or more of the branches of the derivation, or removing it altogether.

Idea of the Proof

We perform induction on the structure of the deduction, via

1. The *cut-rank* of the deduction.

2. The *rank* of the cut formula.

3. The *level* of the cut.

and then proceed using case analysis.

The Level of a cut

The level of a cut is the sum of the heights of the deductions of the premisse(s). For instance, the level of

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma_0 \Rightarrow D\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ D, \Gamma_1 \Rightarrow \Delta\end{matrix}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta}\, Cut$$

is equal to

$$h\left(\mathcal{D}_0\right) + h\left(\mathcal{D}_1\right)$$

where $h\left(\cdot\right)$ is the height of a deduction.

The Rank of a formula

The rank of a formula, A, is $|A| + 1$, where $|A|$ is recursively defined as

$$
\begin{aligned}
|P| &= 0 && \text{for atomic } P \\
|\bot| &= 0 \\
|A \circ B| &= \max(|A|, |B|) + 1 && \text{for binary } \circ \\
|\circ A| &= |A| + 1 && \text{for unary } \circ
\end{aligned}
$$

The Cutrank of a deduction

The **Cutrank** of $\mathcal{D}$, written $cr\,(\mathcal{D})$, is given as

$$max_{A \in \mathcal{D}}\,(rank\,(A))$$

where A is a cut formula in $\mathcal{D}$.

Case analysis

- At least one of the premisses is an instance of an axiom.

- The cutformula is not principal in one of the premisses.

- The cutformula is principal in *both* of the premisses.

The case of axioms – 1

Note: symmetry will do most of the work for us; we only look at either the left or right premiss.

The right premiss has Ax or $L\bot$ as the deduction, and *non-principal*:

$$\frac{\Gamma \Rightarrow D \quad D, \Gamma, P \Rightarrow P}{\Gamma, P \Rightarrow P}\,Cut$$

Solution: Use the conclusion as our new axiom.

The case of axioms – 2

The left premiss is Ax, and the succedent is the *principal* formula:

$$\frac{P, \Gamma \Rightarrow P \quad P, P, \Gamma \Rightarrow \Delta}{\Gamma, P \Rightarrow \Delta}\, Cut$$

Solution: Use closure under contraction on the right premiss, which gives us the conclusion.

The case of axioms – 3a

The right premiss is an instance of $L\bot$, and the cut formula is also principal:

$$\dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma \Rightarrow \bot\end{array} \quad \bot, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\, Cut$$

Solution: Should $\mathcal{D}_0$ end with a rule in which $\bot$ is *principal*, then the left premiss is an instance of $L\bot$, so we can proceed as in case 1 and use the conclusion as a new axiom.

The case of axioms – 3b

Should $\mathcal{D}_0$ end with a rule in which $\bot$ is *non-principal*, and came from a one premiss rule R, say, then we have:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_{00}\\ \Gamma' \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow \bot}\,R \qquad \bot, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,Cut$$

which we transform by permuting the cut above the rule, which is shown on the next slide.

$$\cfrac{\cfrac{\Gamma' \Rightarrow \bot \quad \bot, \Gamma' \Rightarrow \Delta'}{\Gamma' \Rightarrow \Delta'}\,Cut}{\Gamma \Rightarrow \Delta}\,R$$

We can perform this permutation, because the *level* of the cut is lower, although the rank of the cutformula and cutrank of the deduction stay the same.

A similar permutation, with a duplication of the cut, is needed if R was a two premiss rule.

Cutformula is non-principal – Case 2

Suppose that D is not principal in, for instance, the left premiss. The idea is, again, to permute the cut(s) above whichever rule was last applied, and *then* apply the rule. We will show only one case; the rest are almost identical.

Case 2 – Last rule was $L\vee$

The deduction is

$$\dfrac{\dfrac{A, \Gamma \Rightarrow D \quad B, \Gamma \Rightarrow D}{A \vee B, \Gamma \Rightarrow D}\, L\vee \qquad \Gamma, D \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\, Cut$$

which becomes

$$\dfrac{\dfrac{A, \Gamma \Rightarrow D \quad D, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \dfrac{B, \Gamma \Rightarrow D \quad D, \Gamma \Rightarrow \Delta}{B, \Gamma \Rightarrow \Delta}}{A \vee B, \Gamma \Rightarrow \Delta}\, L\vee$$

Case 2 – Last rule was $L\vee$

Why is this admissible?

The *cutrank* of the new deduction is the same as the old, the *rank* of the cut formula is the same (since D is the cut formula in both deductions), but the *level* of each new cut is now lower.

So we can apply the IH to each new cut, and thus create a cut-free deduction.

Case 3 – Principal in both premisses

We consider the different forms of D. One of the following must be true

- $D \equiv D_0 \vee D_1$
- $D \equiv D_0 \wedge D_1$
- $D \equiv D_0 \supset D_1$
- $D \equiv \forall x D_0$
- $D \equiv \exists x D_0$

Many of these are tackled in the same manner, so we will only explicitly prove one subcase. $D \equiv D_0 \vee D_1$ and $D \equiv \exists x D_0$ are particularly easy.

Case 3 – $D \equiv D_0 \wedge D_1$

The original deduction is

$$\dfrac{\dfrac{\Gamma \Rightarrow D_0 \quad \Gamma \Rightarrow D_1}{\Gamma \Rightarrow D_0 \wedge D_1}\, R\wedge \qquad \dfrac{\Gamma, D_0, D_1 \Rightarrow \Delta}{D_0 \wedge D_1, \Gamma \Rightarrow \Delta}\, L\wedge}{\Gamma \Rightarrow \Delta}\, Cut$$

which becomes

$$\dfrac{\Gamma \Rightarrow D_1 \qquad \dfrac{\Gamma \Rightarrow D_0 \quad D_0, D_1, \Gamma \Rightarrow \Delta}{D_1, \Gamma \Rightarrow \Delta}\, Cut}{\Gamma \Rightarrow \Delta}\, Cut$$

Justification for $D \equiv D_0 \wedge D_1$

We have reduced the Cutrank in the previous deduction, since obviously

$$|D_0 \wedge D_1| = \max(|D_0|, |D_1|) + 1 > |D_0|, |D_1|$$

Hence we can apply the induction hypothesis to this new deduction, and we are done.

All cases have been dealt with, and so the proof is complete. For full details, see e.g. [TS00].

A more complex example

Consider the more complicated example of trying to eliminate Cut from

$$\dfrac{\dfrac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \supset B}\, R\supset \qquad \dfrac{A \supset B, \Delta \Rightarrow A \qquad A \supset B, B, \Delta \Rightarrow C}{A \supset B, \Delta \Rightarrow C}\, L\supset}{\Gamma, \Delta \Rightarrow C}\, Cut$$

The usual approach of proceeding as in the case $D \equiv D_0 \supset D_1$ will not work due to the additional $A \supset B$ in the right-most premiss.

But, after a little effort, the deduction on the next slide achieves our goal.

A more complex solution

$$
\dfrac{
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\dfrac{
\dfrac{\Gamma A \Rightarrow B}{\Gamma \Rightarrow A \supset B}
\quad
A \supset B, \Delta \Rightarrow A
}{\Gamma\Delta \Rightarrow A}
\quad
\Gamma A \Rightarrow B
}{\Gamma^2\Delta \Rightarrow B}
\quad
\dfrac{
\dfrac{\Gamma A \Rightarrow B}{\Gamma \Rightarrow A \supset B}
\quad
A \supset B, \Gamma \Rightarrow C
}{B\Gamma\Delta \Rightarrow C}
}{\Gamma^3\Delta^2 \Rightarrow C}
$$

Commas have been elided, except where necessary. A few contractions are also needed.

Justification for the solution

Where we have a cut using $A \supset B$, it is at a lower level than the original deduction, so we can apply the IH to the subdeduction, and eliminate the cut.

Now, the remaining deduction has a lower cutrank, since we only are left with the cut-formulae A and B. So we can apply the IH again, to obtain a cut free deduction.

Hence, this was a valid deduction transformation for the purposes of our proof.

- [DKL06] Dyckhoff R., Kesner D., Lengrand S.: *Strong cut-elimination systems for Hudelmaier's depth-bounded sequent calculus for implicational logic*. (2006)

- [TS00] Troelstra A. S. and Schwichtenberg H.:*Basic Proof Theory*. Cambridge University Press, (2000).

- [SU06] Sørensen M.H. and Urzyczyn P.:*Lectures on the Curry-Howard Isomorphism*. Studies in Logic and the Foundations of Mathematics, Volume 149. Elsevier (2006).