Syntactic Conditions for Invertibility in Sequent Calculi

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Overview

We intend to:

- Give a brief explanation of the problem and why it is important
- Describe a way to “pull apart” the rules of a sequent calculus
- Give some simple, checkable conditions which ensure a rule is invertible

Examples

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \qquad \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}$$

$$\frac{\Gamma, \forall x A, [t/x]A \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta} \qquad \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \supset B, \Delta} \qquad \frac{\Gamma, P, B \Rightarrow C}{\Gamma, P, P \supset B \Rightarrow C}$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \wedge B} \qquad \frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Diamond\Delta, \Delta'} \qquad \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta}$$

Examples: Invertible

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(Meta)formulae

Formulae $A, B, C, \ldots$ are defined by:

- $p, q, \ldots$ atoms
- $\bot$
- $\star_s(\vec{A})$
- Example: $p \supset q \wedge \bot$ is a formula

Metaformulae $\phi, \psi, \gamma \ldots$ are defined by:

- $P, Q, \ldots$ atom variables
- $A, B, C \ldots$ formula variables
- $\bot$
- $\star_s(\vec{\phi})$
- Example: $A \supset P \wedge \bot$ is a metaformula

Rules and Inferences

Rules are displaying using metamultisets and metaformulae:

$$\frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

whereas **inferences** are instances of rules, where the metaformula and metamultisets have been instantiated:

$$\frac{\Gamma \Rightarrow p, q, \Delta}{\Gamma \Rightarrow p \vee q, \Delta}$$

Active and Passive

A metaformula ϕ is **active** for a rule iff

- ϕ cannot be arbitrarily instantiated in an instance of the rule
- ϕ is a submetaformula occurrence of an active metaformula of the rule

A metaformula is **passive** iff it is not active for a rule.
A formula is active (passive) for an inference iff it is the instantiation of an active (passive) metaformula.

Active and Passive: Examples

The active metaformulae...

$$\frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

$$\frac{\Box\Gamma \Rightarrow \phi, \Diamond\Delta}{\Gamma', \Box\Gamma \Rightarrow \Box\phi, \Diamond\Delta, \Delta'}$$

$$\frac{\Gamma, P, \phi \Rightarrow \gamma}{\Gamma, P, P \supset \phi \Rightarrow \gamma}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}$$

Active and Passive: Examples

The active metaformulae are highlighted:

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$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}$$

Principal

When we remove all of the passive metaformulae from a rule, we are left with the **active part** of the rule. A metaformula is **principal** iff it occurs in the conclusion of the active part of a rule.

A rule is a **monoprincipal rule** iff there is only one metaformula occurring in the conclusion of the active part. For instance, the following are monoprincipal rules:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \supset \psi, \Delta} \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

whereas

$$\frac{\Gamma, P, \phi \Rightarrow \gamma}{\Gamma, P, P \supset \phi \Rightarrow \gamma}$$

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Decomposable Rules

Remove all the active metaformula from a rule, leaving the *passive part*. A rule is **decomposable** iff

- *Every* premiss of the passive part is identical
- *Every* premiss of the passive part is a sub-sequent of the conclusion of the passive part

The second condition could be written as: if $\Gamma \Rightarrow \Delta$ was a premiss of the passive part of a rule, and $\Gamma' \Rightarrow \Delta'$ was the conclusion of the passive part of the same rule, then

- $\Gamma \underline{\underline{\in}}^{+} \Gamma'$
- $\Delta \underline{\underline{\in}}^{+} \Delta'$

Decomposable Rules: Examples

The following are decomposable:

$$\frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \supset \psi, \Delta} \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} \qquad \frac{\Gamma, \forall x \phi, [t/x]\phi \Rightarrow \Delta}{\Gamma', \Gamma, \forall x \phi \Rightarrow \Delta, \Delta'}$$

whereas these rules are not:

$$\frac{\Gamma \Rightarrow \phi \quad \Gamma' \Rightarrow \psi}{\Gamma, \Gamma' \Rightarrow \phi \wedge \psi} \qquad \frac{\Gamma, \phi, \phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}$$

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Normal and IW rules

Take the passive part of a decomposable rule. From the second clause of the definition:

- $\Gamma \stackrel{+}{\subseteq} \Gamma'$
- $\Delta \stackrel{+}{\subseteq} \Delta'$

If $\Gamma = \Gamma'$ **and** $\Delta = \Delta'$, we call the rule **normal**. Otherwise, it is an **implicit weakening** rule.

Strong Admissibility, Strong Invertibility

- The rule

$$\frac{S}{S'}$$

is **strongly admissible** in a calculus iff for every n and every derivation of height n of an instance of S there is a derivation of height $\leq n$ of the corresponding instance of S'.

- Suppose $(\vec{P}, C)$ was a rule in a calculus. We call the rule **strongly invertible** iff $\forall P \in \vec{P}$ the rule

$$\frac{C}{P}$$

is strongly admissible.

dp-Weakening

The rule

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is **strongly admissible** in $\mathcal{R}$ if

- There are no context dependent rules in $\mathcal{R}$
- $\mathcal{R}$ contains axioms of the form $\Gamma, P \Rightarrow P, \Delta$

Proof. Routine.

Invertibility

Suppose $\mathcal{R}$ is a calculus containing only decomposable monoprincipal rules. The rule

$$\frac{\Gamma \Rightarrow \Delta, \star_s(\vec{\phi})}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible in $\mathcal{R}$ if $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of *every* rule in $\mathcal{R}$ with $\star_s(\vec{\phi})$ principal on the right.

Invertibility - Proof Sketch

- Proof is by induction on the height of an instance of a derivation with root $\Gamma \Rightarrow \Delta, \star_s(\vec{B})$
- The base case is straightforward
- For the induction step, there are four cases:
  1. $\star_s(\vec{B})$ is principal for the last inference and it was an instance of a normal rule
  2. $\star_s(\vec{B})$ is principal for the last inference and it was an instance of an IW rule
  3. $\star_s(\vec{B})$ is not principal for the last inference and it was an instance of a normal rule
  4. $\star_s(\vec{B})$ is not principal for the last inference and it was an instance of an IW rule

Invertibility - Proof Sketch 1,2

These cases are simple.

1. $\Gamma' \Rightarrow \Delta'$ is a premiss of the active part of the inference, and so $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is derivable at a lower height.
2. The final inference is

$$\frac{\cdots \quad \Gamma_1, \Gamma' \Rightarrow \Delta_1, \Delta' \quad \cdots}{\Gamma_1, \Gamma_2 \Rightarrow \star_s(\vec{B}), \Delta_1, \Delta_2}$$

Weaken the appropriate premiss with the appropriate formulae.

Invertibility - Proof Sketch 3

KEY IDEA:

- $\star_s(\vec{B})$ is not principal for the last inference, THEN
- $\star_s(\vec{B})$ is in the passive part of the inference, THEN
- $\star_s(\vec{B})$ occurs in *every* premiss of the inference, SO
- We can apply the induction hypothesis to every premiss, THEN
- The passive part now contains Γ', Δ' instead of $\star_s(\vec{B})$, but the active part is unchanged, SO
- Apply a new inference with this new passive part.

Invertibility - Proof Sketch 4

The final inference has passive part

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \cdots \quad \Gamma_1 \Rightarrow \Delta_1}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

- If $\star_s(\vec{B})$ was in Δ_1, then the case is the same 3
- If $\star_s(\vec{B})$ was in Δ_2, then we use instead $\Gamma_2 + \Gamma'$ and $(\Delta_2 - \star_s(\vec{B})) + \Delta'$ as the new implicit weakenings

Invertibility - Easy Extensions and Example

It is a simple matter find similar results for

- **Left** rules
- **Single-succedent** calculi (left and right)
- **First-order** calculi

Example: $R\vee$ from **G3cp**

$$\frac{\Gamma \Rightarrow \phi \vee \psi, \Delta}{\Gamma \Rightarrow \phi, \psi, \Delta}$$

is strongly admissible. $\Gamma' = \emptyset$ and $\Delta' = \wr \phi, \psi \int$

Invertibility - Rule sets

Suppose we had a set of monoprincipal normal rules. If each rule had a different principal metaformula, then every rule would be invertible.

Proof. Take an arbitrary rule *R*. Take an arbitrary premiss *P* from *R*. Suppose WLOG that *R* was a right-rule with principal formula ϕ; it is the only rule with ϕ principal on the right. Then, the rule

$$\frac{\Gamma \Rightarrow \phi, \Delta}{P}$$

is strongly admissible in the calculus.

Multiprincipal rules

The conclusion of the active part consists of more than one formula, for example

$$\frac{\Gamma, P, B \Rightarrow \gamma}{\Gamma, P, P \supset \phi \Rightarrow \gamma}$$

Change the conditions of the lemma:

- Any atom metaformula in Γ'' (Δ'') occurs in Γ' (Δ')
- If $\bot$ occurs in Γ'', then $\bot$ occurs in Γ'
- For every $\phi \in \Gamma'', \Delta''$, if ϕ is principal for R, then $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ is a premiss of R
- Then

$$\frac{\Gamma, \Gamma'' \Rightarrow \Delta, \Delta''}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

is strongly admissible

Modal rules

- With a few more restrictions, context-dependent rules can be analysed.
- We have the **prime** part of such rules. For example

$$\frac{\Box\Gamma \Rightarrow \phi, \Diamond\Delta}{\Gamma', \Box\Gamma \Rightarrow \Box\phi, \Diamond\Delta, \Delta'}$$

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Further extensions

Suppose **G3cp** was augmented with the extra rules

$$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

Neither would be invertible, but

$$\frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}$$

would be. The results as presented so far would not show this directly, but it would be another minor tweak to show it was invertible.

Isabelle

- Some of these results have been (approximately) formalised in *Isabelle*
- Instead of giving rules and extracting active parts ($\hat{\ }$), the active parts are given, then extended with context (*)
- The two approaches coincide if only normal rules are used
- If some IW rules are used, then $\mathcal{R} \subseteq \hat{\mathcal{R}}^\star$, but the converse does not hold
- These results could be used to semi-automate Cut admissibility proofs in *Isabelle*

Details

- The details are all in a paper available at *www.cs.st-andrews.ac.uk/$\sim$pc*
- The End
- Any Questions?