Cut Elimination in Sequent Calculi

Peter Chapman
University of St Andrews
pc at *cs.st-and.ac.uk*
www.cs.st-andrews.ac.uk/~pc

A Brief History

- Introduced by Gentzen in 1935

- Much more amenable to metamathematical treatment than natural deduction

- Very useful in proof search, given some restrictions

From Natural Deduction...

- Introduction and Elimination rules for each connective

- eg Given we have assumed A and also B, we can deduce $A \wedge B$. Likewise from an assumption of $A \wedge B$ we can deduce A

$$\frac{A \quad\quad B}{A \wedge B}\, I\wedge \qquad \frac{A \wedge B}{A}\, E_R\wedge$$

...to Sequent Calculus

- We use *contexts* to represent a multiset of assumptions, written as Γ or Δ etc.

- A large arrow "$\Rightarrow$" to separate *antecedents* and *succedents*, and to denote a kind of deduction

$$\frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\, R\wedge \qquad \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}\, L\wedge$$

More Rules

- Roughly speaking, left rules correspond to Elimination rules and right rules to Introduction rules

- For instance, if given any x in some formula A, we could replace it with some y, then we could introduce the universal quantifier

$$\frac{\Gamma \Rightarrow [y/x]A}{\Gamma \Rightarrow \forall x A} \; R\forall$$

More Rules

- If we have that from A we can deduce B, then we can derive $A \supset B$

- Likewise if we can derive A from Γ, and we also derive C from B and Γ, then we can derive C from $A \supset B$ and Γ

$$\frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B} R\supset \qquad \frac{\Gamma \Rightarrow A \quad B, \Gamma \Rightarrow C}{A \supset B, \Gamma \Rightarrow C} L\supset$$

The Cut Rule

- The most striking rule is called *Cut*

- It expresses a form of transitivity; roughly if $A \supset B$ and $B \supset C$ then $A \supset C$

$$\frac{\Gamma \Rightarrow A \quad A, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}\, Cut$$

Cut in Action

A derivation using Cut

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\, R\vee \qquad \dfrac{A, \Gamma \Rightarrow C \quad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\, L\vee}{\Gamma \Rightarrow C}\, Cut$$

- Problem – Given only $\Gamma \Rightarrow C$, the conclusion, we cannot reconstruct the derivation

Problems with Cut

- Because Cut is non-invertible, we cannot "reconstruct" premisses from the conclusion

- This makes Proof Search very difficult

- Ideally we want to be able to eliminate all instances of Cut from a derivation

- We can then implement proof search easily

Ways to Proceed

- Two main ways – *Semantic* and *Constructive* Elimination Proofs

- The latter will be the main focus of this talk

- In Semantic Cut Elimination, we show that the set of provable formulae with the Cut rule is the same as the set without the Cut rule

Criticisms of Semantic Method

For Proof Search, the Semantic proofs of Cut elimination are not very useful.

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\,R\vee \qquad \dfrac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C}\,L\vee}{\Gamma \Rightarrow C}\,Cut$$

We are given no clues as to how to derive $\Gamma \Rightarrow C$ from the premisses without using Cut.

Constructive Cut Elimination

We want a proof that shows how to eliminate each different type of Cut, depending on the Cut formula:

- if the Cut formula is part of an axiom
- if the Cut formula is derived from the immediate premiss(es)
- if the Cut formula is not derived from at least one of the immediate premiss(es)

Overview of Proof Technique

- Use structural induction, based on the Cut formula and some tuple of measures

- Then case analysis

- Often long and repetitive – even Gentzen in his original paper omitted details

A Few More Details

The measures used, typically, are

- *Cut-rank* of the deduction
- *Rank* of the Cut formula
- *Level* of the Cut

in that order. So, if the Cut-rank of the deduction is reduced, the Rank of the formula and the level of the Cut can be increased, and the IH will still apply.

Technical Definitions

- The **Level** of a Cut is the sum of the heights of the (Cut-free) deductions of the premisses

- The **Rank** of a formula is defined recursively as $|A| + 1$ where we have

$$
\begin{array}{rcll}
|P| & = & 0 & \text{for atomic } P \\
|\bot| & = & 0 & \\
|A \circ B| & = & \max(|A|, |B|) + 1 & \text{for binary } \circ \\
|\circ A| & = & |A| + 1 & \text{for unary } \circ
\end{array}
$$

- The **Cut-rank** of a deduction is the maximum of the ranks of all the Cut formulae in the deduction

An Example from G3i

Here is a deduction where the Cut formula $A \wedge B$ is principal in both premisses

$$\dfrac{\dfrac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \qquad \dfrac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C}}{\Gamma \Rightarrow C}$$

which becomes

$$\dfrac{\Gamma \Rightarrow B \qquad \dfrac{\Gamma \Rightarrow A \quad A, B, \Gamma \Rightarrow C}{B, \Gamma \Rightarrow C}}{\Gamma \Rightarrow C}$$

Why this is a valid transformation

We can apply the IH to the new deduction to obtain a Cut-free deduction because the Cut-rank has been reduced.

This is because

$$|A \wedge B| > |A|, |B|$$

which means the rank of the Cut formula has been reduced in each new instance of the Cut rule.

A Summary

- Motivation for Sequent Calculus
- Some rules explained
- Problems with Cut
- Ways to deal with this problem
- A brief example
- Note: Not all calculi admit the Cut rule!