

Dragalin's proofs of cut-admissibility for the intuitionistic sequent calculi **G3i** and **G3i'**

Roy Dyckhoff

School of Mathematical and Computational Sciences
St Andrews University, St Andrews, Scotland

e-mail: rd@dcs.st-and.ac.uk

August 15, 1997

Abstract

Dragalin in his book on *Mathematical Intuitionism* has given an outline proof of the admissibility of Contraction and Cut for the multi-succedent intuitionistic sequent calculus. One can give a corresponding (but simpler) proof for the single-succedent calculus. These proofs are given here in detail as a basis for extension and modification elsewhere.

1 Introduction

Dragalin's proof [1] of admissibility of *Contraction* and *Cut* for a multi-succedent intuitionistic sequent calculus *GHPC* is a useful basis for similar proofs. One can easily construct a similar proof for the single succedent calculus. The proofs are for calculi in which *Contraction* is no longer primitive but an admissible rule. There are hidden complexities: our purpose here is just to give the proofs in detail so that claims elsewhere (e.g. [2], [3]) such as "this proof is like Dragalin's: only the following changes and new cases need to be considered" can be easily verified. Similar proofs are outlined in [4], [5].

We mainly use the terminology and notation of [4] for sequent calculi: thus Dragalin's *GHPC* is here called **G3ip'**. We use $A \supset B$ for implicational formulae and lower case Greek letters for formulae, upper case Greek letters for multisets of formulae. We use $\forall\psi$ for a universal formula, it being understood that ψ is an abstraction $x.\psi(x)$ with instantiations such as ψt (for $\psi(t)$); similarly for existential formulae.

2 Preliminaries

The syntax of formulae and of sequents is standard. The *size* of formulae and the *height* of derivations are calculated as usual.

Inference rules for the single succedent calculus **G3i** are as in [4]. Inference rules for the multi-succedent calculus **G3i'** are as in **3.4.9D** of [4], except that the rule $L\supset$ is

$$\frac{\Gamma, A \supset B \Rightarrow A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \supset B \Rightarrow \Delta} L\supset$$

with no Δ in the left premise.

Lemma 2.1 *The Weakening rules*

$$\frac{\Gamma \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} W \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} W$$

are admissible in **G3i** and in **G3i'**. Weakening does not increase the derivation height.

Proof: Routine. \square

Lemma 2.2 $L\forall$, $L\&$, $L\exists$ and right premiss of $L\supset$ are invertible in **G3i** and in **G3i'**. $R\forall$, $R\&$ and $R\exists$ are invertible in **G3i'**. These inversions do not increase the derivation height.

Proof: Routine. \square

Remark: $L\forall$ does NOT need to be invertible (but is). We shall refer to use of this lemma in derivations as use of the rule *Inv*.

Lemma 2.3 *The EFQ rules*

$$\frac{\Gamma \Rightarrow \perp}{\Gamma \Rightarrow \pi} \qquad \frac{\Gamma \Rightarrow \perp, \Delta}{\Gamma \Rightarrow \Delta}$$

are admissible in **G3i** and **G3i'** respectively.

Proof: By induction on the length of the derivation. \square

Lemma 2.4 *If $\vdash \Pi \Rightarrow \pi$ and x occurs free then we can replace x by another variable without changing the derivation height.*

Proof: Routine. \square

Lemma 2.5 *Suppose the rules R_1, \dots, R_n are admissible. Then any derivation containing instances of these rules can be transformed to one in which they do not occur.*

Proof: By induction on the number of occurrences. For $n = 0$, the result is trivial. For $n > 0$, one randomly picks a topmost occurrence and eliminates it, reducing the derivation to one with $n - 1$ occurrences of the rules; now use the inductive hypothesis. \square

Our general approach to proving admissibility of a rule R is by induction on some measure, here called *rank*. The approach is to assume that there is just one instance of R in a derivation and to show how to replace it by instances of rules already known, or assumed, to be admissible. In practice, this involves replacing a group of inferences ending in the use of R by a group of inferences with the same premises at the top and using rules already known to be admissible and instances of R , but of lower rank. The admissibility of these instances is implied by the induction hypothesis. Lemma 2.5 then shows that their use is eliminable. As noted in [5], it is important to make this strategy clear. One could express the algorithm as the definition of a transformation of derivations, with an innermost-first evaluation strategy. Note that this is much easier than the strong normalisation argument in the Appendix of [1].

3 Admissibility of Contraction in G3i

Theorem 3.1 *The Contraction rule*

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \phi}{\alpha, \Gamma \Rightarrow \phi} \text{Contr}$$

is admissible in **G3i**.

Proof: By induction on size of α , with a subinduction on the height n of the derivation. There are three cases:

1. $n = 0$ and the premise is an axiom:

- (a) $\phi = \alpha$: then $\vdash \alpha, \Gamma \Rightarrow \phi$
- (b) $\phi \in \Gamma$: then $\vdash \alpha, \Gamma \Rightarrow \phi$
- (c) $\perp \in \Gamma$: then $\vdash \alpha, \Gamma \Rightarrow \phi$
- (d) $\perp = \alpha$: then $\vdash \alpha, \Gamma \Rightarrow \phi$.

2. $n > 0$ and α is non-principal:

$$\frac{\alpha, \alpha, \Gamma' \Rightarrow \phi'}{\alpha, \alpha, \Gamma \Rightarrow \phi} R$$

is transformed to

$$\frac{\alpha, \Gamma' \Rightarrow \phi'}{\alpha, \Gamma \Rightarrow \phi} R$$

for a one premise rule R, by inductive hypothesis (same α , shorter derivation). Similarly for two premise rules.

3. $n > 0$ and α is principal:

Case $\alpha = \beta \supset \gamma$

$$\frac{\beta \supset \gamma, \alpha, \Gamma \Rightarrow \beta \quad \gamma, \alpha, \Gamma \Rightarrow \phi}{\alpha, \alpha, \Gamma \Rightarrow \phi} L\supset$$

is transformed to

$$\frac{\frac{\alpha, \alpha, \Gamma \Rightarrow \beta}{\alpha, \Gamma \Rightarrow \beta} \text{Contr} \quad \frac{\frac{\gamma, \alpha, \Gamma \Rightarrow \phi}{\gamma, \gamma, \Gamma \Rightarrow \phi} \text{Inv}}{\gamma, \Gamma \Rightarrow \phi} \text{Contr}}{\alpha, \Gamma \Rightarrow \phi} L\supset$$

Case $\alpha = \beta \& \gamma$

$$\frac{\beta, \gamma, \alpha, \Gamma \Rightarrow \phi}{\beta \& \gamma, \alpha, \Gamma \Rightarrow \phi} L\&$$

is transformed to

$$\frac{\frac{\frac{\beta, \alpha, \gamma, \Gamma \Rightarrow \phi}{\beta, \beta, \gamma, \gamma, \Gamma \Rightarrow \phi} \text{Inv}}{\beta, \beta, \gamma, \Gamma \Rightarrow \phi} \text{Contr}}{\beta, \gamma, \Gamma \Rightarrow \phi} \text{Contr}}{\alpha, \Gamma \Rightarrow \phi} L\&$$

Case $\alpha = \beta \vee \gamma$

$$\frac{\beta, \alpha, \Gamma \Rightarrow \phi \quad \gamma, \alpha, \Gamma \Rightarrow \phi}{\beta \vee \gamma, \alpha, \Gamma \Rightarrow \phi} L\vee$$

is transformed to

$$\frac{\frac{\beta, \alpha, \Gamma \Rightarrow \phi}{\beta, \beta, \Gamma \Rightarrow \phi} Inv \quad \frac{\gamma, \alpha, \Gamma \Rightarrow \phi}{\gamma, \gamma, \Gamma \Rightarrow \phi} Inv}{\frac{\beta, \Gamma \Rightarrow \phi \quad \gamma, \Gamma \Rightarrow \phi}{\beta \vee \gamma, \Gamma \Rightarrow \phi} L\vee} Contr$$

Case $\alpha = \forall \psi$

$$\frac{\forall \psi, \forall \psi, \psi t, \Gamma \Rightarrow \phi}{\forall \psi, \forall \psi, \Gamma \Rightarrow \phi} L\forall$$

is transformed to

$$\frac{\forall \psi, \forall \psi, \psi t, \Gamma \Rightarrow \phi}{\frac{\forall \psi, \psi t, \Gamma \Rightarrow \phi}{\forall \psi, \Gamma \Rightarrow \phi} L\forall} Contr$$

Case $\alpha = \exists \psi$

$$\frac{\exists \psi, \psi x, \Gamma \Rightarrow \phi}{\exists \psi, \exists \psi, \Gamma \Rightarrow \phi} L\exists$$

is transformed to

$$\frac{\frac{\exists \psi, \psi x, \Gamma \Rightarrow \phi}{\psi x, \psi x, \Gamma \Rightarrow \phi} Inv}{\frac{\psi x, \Gamma \Rightarrow \phi}{\exists \psi, \Gamma \Rightarrow \phi} L\exists} Contr$$

□

4 Admissibility of Cut in G3i

Theorem 4.1 *The Cut rule*

$$\frac{\Gamma \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} Cut$$

is admissible.

Proof: By induction on the *rank* of the cut, i.e. the size of the cut formula ϕ and the sum of the heights of the derivations d and d' . By cases:

Case 1 One premise is an axiom

1. $\perp \in \Gamma$: then $\Gamma, \Delta \Rightarrow \pi$ is an axiom
2. $\perp \in \Delta$: then $\Gamma, \Delta \Rightarrow \pi$ is an axiom
3. $\phi \in \Gamma$: then $\vdash \Gamma, \Delta \Rightarrow \pi$ follows from the right premise by *Weakening*
4. $\pi = \phi$: then $\vdash \Gamma, \Delta \Rightarrow \pi$ follows from the left premise by *Weakening*
5. $\pi \in \Delta$: then $\Gamma, \Delta \Rightarrow \pi$ is an axiom
6. $\phi = \perp$: so by Lemma 2.3 $\vdash \Gamma \Rightarrow \pi$, whence $\vdash \Gamma, \Delta \Rightarrow \pi$ by *Weakening*.

Case 2 Neither premise is an axiom and ϕ is not principal in the left premise: by cases according to the rule of the last step of d :

Case $L\supset$ Γ is $\alpha\supset\beta, \Gamma'$

$$\frac{\frac{\alpha\supset\beta, \Gamma' \Rightarrow \alpha \quad \beta, \Gamma' \Rightarrow \phi}{\alpha\supset\beta, \Gamma' \Rightarrow \phi} L\supset \quad \phi, \Delta \Rightarrow \pi}{\alpha\supset\beta, \Gamma', \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\alpha\supset\beta, \Gamma' \Rightarrow \alpha}{\alpha\supset\beta, \Gamma', \Delta \Rightarrow \alpha} Add \quad \frac{\beta, \Gamma' \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\beta, \Gamma', \Delta \Rightarrow \pi} Cut}{\alpha\supset\beta, \Gamma', \Delta \Rightarrow \pi} L\supset$$

Case $L\&$ Γ is $\alpha\&\beta, \Gamma'$

$$[Cut] \frac{\frac{\alpha, \beta, \Gamma' \Rightarrow \phi}{\alpha\&\beta, \Gamma' \Rightarrow \phi} L\& \quad \phi, \Delta \Rightarrow \pi}{\alpha\&\beta, \Gamma', \Delta \Rightarrow \pi}$$

is transformed to

$$\frac{\frac{\alpha, \beta, \Gamma' \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\alpha, \beta, \Gamma', \Delta \Rightarrow \pi} Cut}{\alpha\&\beta, \Gamma', \Delta \Rightarrow \pi} L\&$$

Case $L\vee$ Γ is $\alpha\vee\beta, \Gamma'$

$$\frac{\frac{\alpha, \Gamma' \Rightarrow \phi \quad \beta, \Gamma' \Rightarrow \phi}{\alpha\vee\beta, \Gamma' \Rightarrow \phi} L\vee \quad \phi, \Delta \Rightarrow \pi}{\alpha\vee\beta, \Gamma', \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\alpha, \Gamma' \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\alpha, \Gamma', \Delta \Rightarrow \pi} Cut \quad \frac{\beta, \Gamma' \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\beta, \Gamma', \Delta \Rightarrow \pi} Cut}{\alpha\vee\beta, \Gamma', \Delta \Rightarrow \pi} L\vee$$

Case $L\forall$ Γ is $\forall\psi, \Gamma'$

$$\frac{\frac{\forall\psi, \psi t, \Gamma' \Rightarrow \phi}{\forall\psi, \Gamma' \Rightarrow \phi} L\forall \quad \phi, \Delta \Rightarrow \pi}{\forall\psi, \Gamma', \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\forall\psi, \psi t, \Gamma' \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\forall\psi, \psi t, \Gamma', \Delta \Rightarrow \pi} L\forall}{\forall\psi, \Gamma', \Delta \Rightarrow \pi} Cut$$

Case $L\exists$ Γ is $\exists\psi, \Gamma'$

$$\frac{\frac{\frac{\psi x, \Gamma' \Rightarrow \phi}{\exists\psi, \Gamma' \Rightarrow \phi} L\exists \quad \phi, \Delta \Rightarrow \pi}{\exists\psi, \Gamma', \Delta \Rightarrow \pi} Cut}{\exists\psi, \Gamma', \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\psi x, \Gamma' \Rightarrow \phi \quad \phi, \Delta \Rightarrow \pi}{\psi x, \Gamma', \Delta \Rightarrow \pi} L\exists}{\exists\psi, \Gamma', \Delta \Rightarrow \pi} Cut$$

Case 3 Neither premise is an axiom and ϕ is principal in left premise, but not principal in right premise: by cases according to the last rule used in d' :

Case $L\supset$ Δ is $\alpha\supset\beta, \Delta'$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\frac{\phi, \alpha\supset\beta, \Delta' \Rightarrow \alpha \quad \phi, \beta, \Delta' \Rightarrow \pi}{\phi, \alpha\supset\beta, \Delta' \Rightarrow \pi} L\supset}{\Gamma, \alpha\supset\beta, \Delta' \Rightarrow \pi} Cut}{\Gamma, \alpha\supset\beta, \Delta' \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \phi \quad \phi, \alpha\supset\beta, \Delta' \Rightarrow \alpha}{\Gamma, \alpha\supset\beta, \Delta' \Rightarrow \alpha} Cut \quad \frac{\Gamma \Rightarrow \phi \quad \phi, \beta, \Delta' \Rightarrow \pi}{\Gamma, \beta, \Delta' \Rightarrow \pi} L\supset}{\Gamma, \alpha\supset\beta, \Delta' \Rightarrow \pi} Cut$$

Case $R\supset$ $\pi = \alpha\supset\beta$

$$\frac{\frac{\Gamma \Rightarrow \phi \quad \frac{\alpha, \phi, \Delta \Rightarrow \beta}{\phi, \Delta \Rightarrow \alpha\supset\beta} R\supset}{\Gamma, \Delta \Rightarrow \alpha\supset\beta} Cut}{\Gamma, \Delta \Rightarrow \alpha\supset\beta} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \phi \quad \alpha, \phi, \Delta \Rightarrow \beta}{\alpha, \Gamma, \Delta \Rightarrow \beta} Cut}{\Gamma, \Delta \Rightarrow \alpha\supset\beta} L\supset$$

Case $L\&$ Δ is $\alpha\&\beta, \Delta'$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \alpha, \beta, \Delta' \Rightarrow \pi}{\phi, \alpha\&\beta, \Delta' \Rightarrow \pi} L\&}{\Gamma, \alpha\&\beta, \Delta' \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \phi \quad \phi, \alpha, \beta, \Delta' \Rightarrow \pi}{\Gamma, \alpha, \beta, \Delta' \Rightarrow \pi} Cut}{\Gamma, \alpha\&\beta, \Delta' \Rightarrow \pi} L\&$$

Case $R\&$ $\pi = \alpha\&\beta$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \Delta \Rightarrow \alpha \quad \phi, \Delta \Rightarrow \beta}{\phi, \Delta \Rightarrow \alpha\&\beta} R\&}{\Gamma, \Delta \Rightarrow \alpha\&\beta} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \phi \quad \phi, \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha} Cut \quad \frac{\Gamma \Rightarrow \phi \quad \phi, \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \beta} Cut}{\Gamma, \Delta \Rightarrow \alpha\&\beta} R\&$$

Case LV Δ is $\alpha \vee \beta, \Delta'$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \alpha, \Delta' \Rightarrow \pi \quad \phi, \beta, \Delta' \Rightarrow \pi}{\phi, \alpha \vee \beta, \Delta' \Rightarrow \pi} LV}{\Gamma, \alpha \vee \beta, \Delta' \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \phi \quad \phi, \alpha, \Delta' \Rightarrow \pi}{\Gamma, \alpha, \Delta' \Rightarrow \pi} Cut \quad \frac{\Gamma \Rightarrow \phi \quad \phi, \beta, \Delta' \Rightarrow \pi}{\Gamma, \beta, \Delta' \Rightarrow \pi} Cut}{\Gamma, \alpha \vee \beta, \Delta' \Rightarrow \pi} LV$$

Case RV $\pi = \alpha \vee \beta$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \Delta \Rightarrow \alpha}{\phi, \Delta \Rightarrow \alpha \vee \beta} RV}{\Gamma, \Delta \Rightarrow \alpha \vee \beta} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \phi \quad \phi, \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha} Cut}{\Gamma, \Delta \Rightarrow \alpha \vee \beta} RV$$

Case $L\forall$ Δ is $\forall\psi, \Delta'$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\psi, \forall\psi, \psi t, \Delta' \Rightarrow \pi}{\phi, \forall\psi, \Delta' \Rightarrow \pi} L\forall}{\Gamma, \forall\psi, \Delta' \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \forall\psi, \psi t, \Delta' \Rightarrow \pi}{\Gamma, \forall\psi, \psi t, \Delta' \Rightarrow \pi} L\forall}{\Gamma, \forall\psi, \Delta' \Rightarrow \pi} Cut$$

Case $R\forall$ $\pi = \forall\psi$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \Delta \Rightarrow \psi x}{\phi, \Delta \Rightarrow \forall\psi} R\forall}{\Gamma, \Delta \Rightarrow \forall\psi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \Delta \Rightarrow \psi x}{\Gamma, \Delta \Rightarrow \psi x} R\forall}{\Gamma, \Delta \Rightarrow \forall\psi} Cut$$

Case $L\exists$ Δ is $\exists\psi, \Delta'$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \psi x, \Delta' \Rightarrow \pi}{\phi, \exists\psi, \Delta' \Rightarrow \pi} L\exists}{\Gamma, \exists\psi, \Delta' \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \psi x, \Delta' \Rightarrow \pi}{\Gamma, \psi x, \Delta' \Rightarrow \pi} L\exists}{\Gamma, \exists\psi, \Delta' \Rightarrow \pi} Cut$$

Case $R\exists$ $\pi = \exists\psi$

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \Delta \Rightarrow \psi t}{\phi, \Delta \Rightarrow \exists\psi} R\exists}{\Gamma, \Delta \Rightarrow \exists\psi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \phi \quad \frac{\phi, \Delta \Rightarrow \psi t}{\Gamma, \Delta \Rightarrow \psi t} R\exists}{\Gamma, \Delta \Rightarrow \exists\psi} Cut$$

Case 4 ϕ is principal in both premises: by cases according to the form of ϕ :

Case $\phi = \alpha \supset \beta$

$$\frac{\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \supset \beta} R\supset \quad \frac{\alpha \supset \beta, \Delta \Rightarrow \alpha \quad \beta, \Delta \Rightarrow \pi}{\alpha \supset \beta, \Delta \Rightarrow \pi} L\supset}{\Gamma, \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \alpha \supset \beta \quad \alpha \supset \beta, \Delta \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha} Cut \quad \frac{\alpha, \Gamma \Rightarrow \beta \quad \beta, \Delta \Rightarrow \pi}{\alpha, \Gamma, \Delta \Rightarrow \pi} Cut}{\frac{\Gamma, \Delta, \Gamma, \Delta \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} Contr} Cut$$

Case $\phi = \alpha \& \beta$

$$\frac{\frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \& \beta} R\& \quad \frac{\alpha, \beta, \Delta \Rightarrow \pi}{\alpha \& \beta, \Delta \Rightarrow \pi} L\&}{\Gamma, \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \alpha \quad \frac{\Gamma \Rightarrow \beta \quad \alpha, \beta, \Delta \Rightarrow \pi}{\alpha, \Gamma, \Delta \Rightarrow \pi} Cut}{\frac{\Gamma, \Gamma, \Delta \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} Contr} Cut$$

Case $\phi = \alpha \vee \beta$

$$\frac{\frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} R\vee \quad \frac{\alpha, \Delta \Rightarrow \pi \quad \beta, \Delta \Rightarrow \pi}{\alpha \vee \beta, \Delta \Rightarrow \pi} L\vee}{\Gamma, \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} Cut$$

Case $\phi = \forall \psi$

$$\frac{\frac{\Gamma \Rightarrow \psi x}{\Gamma \Rightarrow \forall \psi} R\forall \quad \frac{\forall \psi, \psi t, \Delta \Rightarrow \pi}{\forall \psi, \Delta \Rightarrow \pi} L\forall}{\Gamma, \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \psi t \quad \frac{\Gamma \Rightarrow \forall \psi \quad \forall \psi, \psi t, \Delta \Rightarrow \pi}{\Gamma, \psi t, \Delta \Rightarrow \pi} Cut}{\frac{\Gamma, \Gamma, \Delta \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} Contr} Cut$$

where the leftmost premise is obtained by substitution of t for the new variable x in $\Gamma \Rightarrow \psi x$.

Case $\phi = \exists\psi$

$$\frac{\frac{\Gamma \Rightarrow \psi t}{\Gamma \Rightarrow \exists\psi} R\exists \quad \frac{\psi x, \Delta \Rightarrow \pi}{\exists\psi, \Delta \Rightarrow \pi} L\exists}{\Gamma, \Delta \Rightarrow \pi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \psi t \quad \psi t \Delta \Rightarrow \pi}{\Gamma, \Delta \Rightarrow \pi} Cut$$

where the right premise is obtained by substitution of t for the new variable x in $\psi x, \Delta \Rightarrow \pi$.

□

5 Admissibility of Contraction in $\mathbf{G3i'}$

Theorem 5.1 *The Contraction rules*

$$\frac{\Gamma, \alpha, \alpha \Rightarrow \Delta}{\Gamma, \alpha \Rightarrow \Delta} Contr-L \quad \frac{\Gamma \Rightarrow \alpha, \alpha, \Delta}{\Gamma \Rightarrow \alpha, \Delta} Contr-R$$

are admissible.

Proof: The *Contr-R* rule is easily dealt with: we consider only *Contr-L*. By induction on size of α , with a subinduction on the height n of the derivation. There are three cases:

1. $n = 0$ and the premise is an axiom:

- (a) α is an atomic formula in Δ : then $\vdash \alpha, \Gamma \Rightarrow \Delta$
- (b) Γ meets Δ in an atomic formula: then $\vdash \alpha, \Gamma \Rightarrow \Delta$
- (c) $\perp \in \Gamma$: then $\vdash \alpha, \Gamma \Rightarrow \Delta$
- (d) $\perp = \alpha$: then $\vdash \alpha, \Gamma \Rightarrow \Delta$.

2. $n > 0$ and α is non-principal:

$$\frac{\alpha, \alpha, \Gamma' \Rightarrow \Delta'}{\alpha, \alpha, \Gamma \Rightarrow \Delta} R$$

is transformed to

$$\frac{\alpha, \Gamma' \Rightarrow \Delta'}{\alpha, \Gamma \Rightarrow \Delta} R$$

for a one premise rule R , by inductive hypothesis (same α , shorter derivation). Similarly for two premise rules.

3. $n > 0$ and α is principal:

Case $\alpha = \beta \supset \gamma$

$$\frac{\beta \supset \gamma, \alpha, \Gamma \Rightarrow \beta \quad \gamma, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \alpha, \Gamma \Rightarrow \Delta} L\supset$$

is transformed to

$$\frac{\frac{\alpha, \alpha, \Gamma \Rightarrow \beta}{\alpha, \Gamma \Rightarrow \beta} Contr-L \quad \frac{\frac{\gamma, \alpha, \Gamma \Rightarrow \Delta}{\gamma, \gamma, \Gamma \Rightarrow \Delta} Inv \quad \frac{\gamma, \gamma, \Gamma \Rightarrow \Delta}{\gamma, \Gamma \Rightarrow \Delta} Contr-L}{\alpha, \Gamma \Rightarrow \Delta} L\supset$$

Case $\alpha = \beta \& \gamma$

$$\frac{\beta, \gamma, \alpha, \Gamma \Rightarrow \Delta}{\beta \& \gamma, \alpha, \Gamma \Rightarrow \Delta} L\&$$

is transformed to

$$\frac{\frac{\frac{\frac{\beta, \alpha, \gamma, \Gamma \Rightarrow \Delta}{\beta, \beta, \gamma, \gamma, \Gamma \Rightarrow \Delta} Inv}{\beta, \beta, \gamma, \Gamma \Rightarrow \Delta} Contr-L}{\beta, \gamma, \Gamma \Rightarrow \Delta} Contr-L}{\alpha, \Gamma \Rightarrow \Delta} L\&$$

Case $\alpha = \beta \vee \gamma$

$$\frac{\beta, \alpha, \Gamma \Rightarrow \Delta \quad \gamma, \alpha, \Gamma \Rightarrow \Delta}{\beta \vee \gamma, \alpha, \Gamma \Rightarrow \Delta} LV$$

is transformed to

$$\frac{\frac{\frac{\beta, \alpha, \Gamma \Rightarrow \Delta}{\beta, \beta, \Gamma \Rightarrow \Delta} Inv}{\beta, \Gamma \Rightarrow \Delta} Contr-L \quad \frac{\frac{\gamma, \alpha, \Gamma \Rightarrow \Delta}{\gamma, \gamma, \Gamma \Rightarrow \Delta} Inv}{\gamma, \Gamma \Rightarrow \Delta} Contr-L}{\beta \vee \gamma, \Gamma \Rightarrow \Delta} LV$$

Case $\alpha = \forall \psi$

$$\frac{\forall \psi, \forall \psi, \psi t, \Gamma \Rightarrow \Delta}{\forall \psi, \forall \psi, \Gamma \Rightarrow \Delta} L\forall$$

is transformed to

$$\frac{\frac{\forall \psi, \forall \psi, \psi t, \Gamma \Rightarrow \Delta}{\forall \psi, \psi t, \Gamma \Rightarrow \Delta} Contr-L}{\forall \psi, \Gamma \Rightarrow \Delta} L\forall$$

Case $\alpha = \exists \psi$

$$\frac{\exists \psi, \psi x, \Gamma \Rightarrow \Delta}{\exists \psi, \exists \psi, \Gamma \Rightarrow \Delta} L\exists$$

is transformed to

$$\frac{\frac{\frac{\exists \psi, \psi x, \Gamma \Rightarrow \Delta}{\psi x, \psi x \Gamma \Rightarrow \Delta} Inv}{\psi x, \Gamma \Rightarrow \Delta} Contr-L}{\exists \psi, \Gamma \Rightarrow \Delta} L\exists$$

□

6 Admissibility of Cut in G3i'

Theorem 6.1 *The Cut rule*

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is admissible.

Proof: By induction on the *rank* of the cut, i.e. the size s of the cut formula, the height n of the derivation of the first premise and the height n' of the derivation of the second premise. We indicate when generating new cuts an upper bound, e.g. $(s-1, X, X)$ or $(s, n-1, X)$, on the rank of the new cut(s), where X indicates a “don’t care” quantity. (In fact, we could use a simpler notion of rank, as in 4.1: but we follow [1].) For each rank, by cases:

Case 1: One of the premises of the cut is an axiom:

Case Left: The left premise is an axiom:

1. ϕ is an atomic formula in Γ : then $\vdash \Gamma, \Pi \Rightarrow \Delta, \Phi$ follows from $\vdash \phi, \Pi \Rightarrow \Phi$ by *Weakening*
2. Γ, Δ have an atomic formula in common: then $\Gamma, \Pi \Rightarrow \Delta, \Phi$ is an axiom
3. $\perp \in \Gamma$: then $\Gamma, \Pi \Rightarrow \Delta, \Phi$ is an axiom.

Case Right: The right premise is an axiom:

1. ϕ is an atomic formula in Φ : then $\vdash \Gamma, \Pi \Rightarrow \Delta, \Phi$ follows from $\vdash, \Gamma, \Rightarrow \Delta, \phi$ by *Weakening*
2. Π, Φ have an atomic formula in common: then $\Gamma, \Pi \Rightarrow \Delta, \Phi$ is an axiom
3. $\perp \in \Pi$: then $\Gamma, \Pi \Rightarrow \Delta, \Phi$ is an axiom
4. $\phi = \perp$: then $\vdash \Gamma \Rightarrow \Delta$ by Lemma 2.3, whence $\vdash \Gamma, \Pi \Rightarrow \Delta, \Phi$ by *Weakening*.

Case 2: The left premise is obtained by a rule R of which ϕ is not the principal formula. We proceed according to the possible cases of R :

Case $L\supset$: $\Gamma = \alpha\supset\beta, \Gamma'$

$$\frac{\frac{\alpha\supset\beta, \Gamma' \Rightarrow \alpha \quad \beta, \Gamma' \Rightarrow \Delta, \phi}{\alpha\supset\beta, \Gamma' \Rightarrow \Delta\phi} L\supset \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\alpha\supset\beta, \Gamma' \Rightarrow \alpha}{\alpha\supset\beta, \Gamma', \Pi \Rightarrow \alpha} W \quad \frac{\beta, \Gamma' \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Phi}{\beta, \Gamma', \Pi \Rightarrow \Delta, \Phi} L\supset}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

with one cut of rank $\leq (s, n-1, n')$.

Case $R\supset$: $\Delta = \alpha\supset\beta, \Delta'$

$$\frac{\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\supset\beta, \Delta', \phi} R\supset \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\alpha\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha\supset\beta} R\supset}{\Gamma, \Pi \Rightarrow \Delta, \Phi} W$$

with no cuts.

Case $L\&$: $\Gamma = \alpha\&\beta, \Gamma'$

$$\frac{\frac{\alpha, \beta, \Gamma' \Rightarrow \Delta, \phi}{\alpha\&\beta, \Gamma' \Rightarrow \Delta, \phi} L\& \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\alpha, \beta, \Gamma' \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Phi}{\alpha, \beta, \Gamma', \Pi \Rightarrow \Delta, \Phi} Cut}{\alpha\&\beta, \Gamma' \Pi \Rightarrow \Delta, \Phi} L\&$$

with one cut of rank $\leq (s, n - 1, n')$.

Case $R\&$: $\Delta = \alpha\&\beta, \Delta'$

$$\frac{\frac{\Gamma \Rightarrow \alpha, \Delta', \phi \quad \Gamma \Rightarrow \beta, \Delta', \phi}{\Gamma \Rightarrow \alpha\&\beta, \Delta', \phi} R\& \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \alpha, \Delta', \phi \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \alpha, \Delta', \Phi} Cut \quad \frac{\Gamma \Rightarrow \beta, \Delta', \phi \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \beta, \Delta', \Phi} R\&}{\Gamma, \Pi \Rightarrow \alpha\&\beta, \Delta', \Phi} Cut$$

with two cuts of rank $\leq (s, n - 1, n')$ and $(s, n - 1, n')$.

Case $L\vee$: $\Gamma = \alpha \vee \beta, \Gamma'$

$$\frac{\frac{\alpha, \Gamma' \Rightarrow \Delta, \phi \quad \beta, \Gamma' \Rightarrow \Delta, \phi}{\alpha \vee \beta, \Gamma' \Rightarrow \Delta, \phi} L\vee \quad \phi, \Pi \Rightarrow \Phi}{\alpha \vee \beta, \Gamma', \Pi \Rightarrow \Delta, \phi} Cut$$

is transformed to

$$\frac{\frac{\alpha, \Gamma' \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Phi}{\alpha, \Gamma', \Pi \Rightarrow \Delta, \Phi} Cut \quad \frac{\beta, \Gamma' \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Phi}{\beta, \Gamma', \Pi \Rightarrow \Delta, \Phi} Cut}{\alpha \vee \beta, \Gamma', \Pi \Rightarrow \Delta, \Phi} L\vee$$

with two cuts of rank $\leq (s, n - 1, n')$ and $(s, n - 1, n')$.

Case $R\vee$: $\Delta = \alpha \vee \beta, \Delta'$

$$\frac{\frac{\Gamma \Rightarrow \alpha, \beta, \Delta', \phi}{\Gamma \Rightarrow \alpha \vee \beta, \Delta', \phi} R\vee \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \alpha \vee \beta, \Delta', \Phi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \alpha, \beta, \Delta', \phi \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \alpha, \beta, \Delta', \Phi} Cut}{\Gamma, \Pi \Rightarrow \alpha \vee \beta, \Delta', \Phi} R\vee$$

with one cut of rank $\leq (s, n - 1, n')$.

Case $L\forall$: $\Gamma = \forall\psi, \Gamma'$

$$\frac{\frac{\forall\psi, \psi t, \Gamma' \Rightarrow \Delta, \phi}{\forall\psi, \Gamma' \Rightarrow \Delta, \phi} L\forall \quad \phi, \Pi \Rightarrow \Phi}{\forall\psi, \Gamma' \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\forall\psi, \psi t, \Gamma' \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \Phi}{\forall\psi, \psi t, \Gamma', \Pi \Rightarrow \Delta, \Phi} Cut}{\forall\psi, \Gamma', \Pi \Rightarrow \Delta, \Phi} L\forall$$

with one cut of rank $\leq (s, n - 1, n')$.

Case $R\forall$: $\Delta = \forall\psi, \Delta'$

$$\frac{\frac{\Gamma \Rightarrow \psi x}{\Gamma \Rightarrow \forall\psi, \Delta', \phi} R\forall \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \forall\psi, \Delta', \Phi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \psi x}{\Gamma \Rightarrow \forall\psi} R\forall}{\Gamma, \Pi \Rightarrow \forall\psi, \Delta', \Pi} WW$$

with no cuts.

Case $L\exists$: $\Gamma = \exists\psi, \Gamma'$

$$\frac{\frac{\psi x, \Gamma' \Rightarrow \Delta, \phi}{\exists\psi, \Gamma' \Rightarrow \Delta, \phi} L\exists \quad \phi, \Pi \Rightarrow \Phi}{\exists\psi, \Gamma', \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\psi x, \Gamma' \Rightarrow \Delta, \Phi \quad \phi, \Pi \Rightarrow \Phi}{\psi x, \Gamma', \Pi \Rightarrow \Delta, \Phi} Cut}{\exists\psi, \Gamma', \Pi \Rightarrow \Delta, \Phi} L\exists$$

with one cut, of rank $\leq (s, n - 1, n')$: since x is not free in Γ', Δ, ϕ , we can rename free occurrences of x in $\phi, \Pi \Rightarrow \Phi$ to a variable other than x without changing ϕ . So x is w.l.o.g. not free in $\Gamma', \Pi \Rightarrow \Delta, \Phi$, so final use of $L\exists$ is permitted.

Case $R\exists$: $\Delta = \exists\psi, \Delta'$

$$\frac{\frac{\Gamma \Rightarrow \exists\psi, \psi t, \Delta', \phi}{\Gamma \Rightarrow \exists\psi, \Delta', \phi} R\exists \quad \phi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \exists\psi, \Delta', \Phi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \exists \psi, \psi t, \Delta', \phi \quad \phi, \Pi \Rightarrow \Phi}{\frac{\Gamma \Rightarrow \exists \psi, \psi t, \Delta', \phi}{\Gamma, \Pi \Rightarrow \exists \psi, \Delta', \Phi} R\exists} Cut$$

with one cut, of rank $\leq (s, n - 1, n')$.

Case 3: The right premise is obtained by a rule R of which ϕ is not the principal formula and the left premise is obtained by a rule of which ϕ is the principal formula, by cases on R :

Case $L\supset$: $\Pi = \alpha \supset \beta, \Pi'$. The obvious translation of

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \alpha \supset \beta, \Pi' \Rightarrow \alpha \quad \phi, \beta, \Pi' \Rightarrow \Phi}{\phi, \alpha \supset \beta, \Pi' \Rightarrow \Phi} L\supset}{\Gamma, \alpha \supset \beta, \Pi' \Rightarrow \Delta, \Phi} Cut$$

to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \alpha \supset \beta, \Pi' \Rightarrow \alpha}{\Gamma, \alpha \supset \beta, \Pi' \Rightarrow \Delta, \alpha} Cut \quad \frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \beta, \Pi' \Rightarrow \Phi}{\beta, \Gamma, \Pi' \Rightarrow \Delta, \Phi} L\supset}{\Gamma, \alpha \supset \beta, \Pi' \Rightarrow \Delta, \Phi} Cut$$

does not work, because of the Δ in the first premise of the $L\supset$ inference. But, by assumption, ϕ is the principal formula of the inference leading to the left premise of the original Cut . If this inference is by $L\supset$ or $L\forall$, then from its premise we may derive just $\Gamma \supset \phi$, and now the translation above works as if Δ was empty, with two cuts of rank $\leq (s, n, n' - 1)$; then we follow it by weakening the conclusion with Δ . Otherwise, the inference is by one of $R\&$, $R\vee$ or $R\exists$ and the corresponding left rule is invertible. For example, if $\phi = \psi \vee \theta$, we have

$$\frac{\frac{\Gamma \Rightarrow \Delta, \psi, \theta}{\Gamma \Rightarrow \Delta, \psi \vee \theta} R\vee \quad \frac{\dots \dots}{\psi \vee \theta, \alpha \supset \beta, \Pi' \Rightarrow \Phi} L\supset}{\Gamma, \alpha \supset \beta, \Pi' \Rightarrow \Delta, \Phi} Cut$$

which is transformed to

$$\frac{\frac{\frac{\psi \vee \theta, \alpha \supset \beta, \Pi' \Rightarrow \Phi}{\psi, \alpha \supset \beta, \Pi' \Rightarrow \Phi} Inv \quad \frac{\psi \vee \theta, \alpha \supset \beta, \Pi' \Rightarrow \Phi}{\theta, \alpha \supset \beta, \Pi' \Rightarrow \Phi} Inv}{\Gamma, \alpha \supset \beta, \Pi' \Rightarrow \Delta, \Phi, \theta} Cut \quad \frac{\psi \vee \theta, \alpha \supset \beta, \Pi' \Rightarrow \Phi}{\theta, \alpha \supset \beta, \Pi' \Rightarrow \Phi} Inv}{\frac{\Gamma, \alpha \supset \beta, \Pi', \alpha \supset \beta, \Pi' \Rightarrow \Delta, \Phi, \Phi}{\Gamma, \alpha \supset \beta, \Pi' \Rightarrow \Delta, \Phi} Contr} Cut$$

with cuts of rank $\leq (s - 1, X, X)$.

Case $R\supset$: $\Phi = \alpha \supset \beta, \Phi'$. The derivation contains

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\alpha, \phi, \Pi \Rightarrow \beta}{\phi, \Pi \Rightarrow \alpha \supset \beta, \Phi'} R\supset}{\Gamma, \Pi \Rightarrow \Delta, \alpha \supset \beta, \Phi'} Cut} Cut$$

and again, the obvious translation to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \alpha, \phi, \Pi \Rightarrow \beta}{\alpha, \Gamma, \Pi \Rightarrow \Delta, \beta} \text{Cut}}{\Gamma, \Pi \Rightarrow \Delta, \alpha \supset \beta, \Phi'} R\supset$$

does not work, because of Δ in the premiss of $R\supset$. But, by assumption, the left premiss is the conclusion of a rule with ϕ the principal formula. If this rule is $R\supset$ or $R\forall$, we may derive instead just $\Gamma \Rightarrow \phi$, and now the 'obvious' translation works with Δ absent, and one cut of rank $\leq (s, n, n' - 1)$; then we follow it by weakening with Δ . In the case of any other rule, i.e. one of $R\&$, $R\forall$, and $R\exists$, we note that the corresponding left rule is invertible. For example, if $\phi = \psi \vee \theta$ we have

$$\frac{\frac{\Gamma \Rightarrow \Delta, \psi, \theta}{\Gamma \Rightarrow \Delta, \psi \vee \theta} R\forall \quad \phi, \Pi \Rightarrow \alpha \supset \beta, \Phi'}{\Gamma, \Pi \Rightarrow \Delta, \alpha \supset \beta, \Phi'} \text{Cut}$$

which is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \psi, \theta \quad \frac{\psi \vee \theta, \Pi \Rightarrow \Phi}{\psi, \Pi \Rightarrow \Phi} \text{Inv}}{\Gamma, \Pi \Rightarrow \Delta, \theta, \Phi} \text{Cut} \quad \frac{\psi \vee \theta, \Pi \Rightarrow \Phi}{\theta, \Pi \Rightarrow \Phi} \text{Inv}}{\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Phi, \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Contr}} \text{Cut}$$

with cuts of rank $\leq (s - 1, X, X)$.

Case $L\&$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \alpha, \beta, \Pi' \Rightarrow \Phi}{\phi, \alpha \& \beta, \Pi' \Rightarrow \Phi} L\&}{\alpha \& \beta, \Gamma, \Pi' \Rightarrow \Delta, \Phi} \text{Cut}}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \alpha, \beta, \Pi' \Rightarrow \Phi}{\Gamma, \alpha, \beta, \Pi' \Rightarrow \Delta, \Phi} \text{Cut}}{\Gamma, \alpha \& \beta, \Pi' \Rightarrow \Delta, \Phi} L\&$$

with one cut of rank $\leq (s, n, n' - 1)$.

Case $R\&$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \Pi \Rightarrow \alpha, \Phi' \quad \Phi, \Pi \Rightarrow \beta, \Phi'}{\phi, \Pi \Rightarrow \alpha \& \beta, \Phi'} R\&}{\Gamma, \Pi \Rightarrow \Delta, \alpha \& \beta, \Phi'} \text{Cut}}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \alpha, \Phi'}{\Gamma, \Pi \Rightarrow \alpha, \Delta, \Phi'} \text{Cut} \quad \frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \beta, \Phi'}{\Gamma, \Pi \Rightarrow \beta, \Delta, \Phi'} \text{Cut}}{\Gamma, \Pi \Rightarrow \alpha \& \beta, \Delta, \Phi'} R\&$$

with two cuts of rank $\leq (s, n, n' - 1)$.

Case $L\vee$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \alpha, \Pi' \Rightarrow \Phi \quad \phi, \beta, \Pi' \Rightarrow \Phi}{\phi, \alpha \vee \beta, \Pi' \Rightarrow \Phi} L\vee}{\Gamma, \alpha \vee \beta, \Pi' \Rightarrow \Delta, \Phi} Cut}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \alpha, \Pi' \Rightarrow \Phi}{\Gamma, \alpha, \Pi' \Rightarrow \Delta, \Phi} Cut \quad \frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \beta, \Pi' \Rightarrow \Phi}{\Gamma, \beta, \Pi' \Rightarrow \Delta, \Phi} Cut}{\Gamma, \alpha \vee \beta, \Pi' \Rightarrow \Delta, \Phi} L\vee}$$

with two cuts of rank $\leq (s, n, n' - 1)$.

Case $R\vee$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \Pi \Rightarrow \alpha, \beta, \Phi'}{\phi, \Pi \Rightarrow \alpha \vee \beta, \Phi'} R\vee}{\Gamma, \Pi \Rightarrow \alpha \vee \beta, \Delta, \Phi'} Cut}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \alpha, \beta, \Phi'}{\Gamma, \Pi \Rightarrow \alpha, \beta, \Delta, \Phi'} Cut}{\Gamma, \Pi \Rightarrow \alpha \vee \beta, \Delta, \Phi'} R\vee}$$

with one cut of rank $\leq (s, n, n' - 1)$.

Case $L\forall$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \forall \psi, \psi t, \Pi' \Rightarrow \Phi}{\phi, \forall \psi, \Pi' \Rightarrow \Phi} L\forall}{\Gamma, \forall \psi, \Pi' \Rightarrow \Delta, \Phi} Cut}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \forall \psi, \psi t, \Pi' \Rightarrow \Phi}{\Gamma, \forall \psi, \psi t, \Pi' \Rightarrow \Delta, \Phi} Cut}{\Gamma, \forall \psi, \Pi' \Rightarrow \Delta, \Phi} L\forall}$$

with one cut of rank $\leq (s, n, n' - 1)$.

Case $R\forall$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \Pi \Rightarrow \psi x}{\phi, \Pi \Rightarrow \forall \psi, \Phi'} R\forall}{\Gamma, \Pi \Rightarrow \Delta, \forall \psi, \Phi'} Cut}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \psi x}{\Gamma, \Pi \Rightarrow \Delta, \psi x} Cut}{\Gamma, \Pi \Rightarrow \Delta, \forall \psi, \Phi'} R\forall}$$

provided Δ is empty, and renaming occurrences of x in Γ where necessary, with one cut of rank $\leq (s, n, n' - 1)$. As before, we may assume ϕ is the principal formula of the final inference in the derivation of the left premise. If that is by the rule $R\supset$ or $r\forall$, we may assume that its conclusion has Δ empty. Otherwise, the rule is $R\&$,

$R\forall$ or $R\exists$ and the corresponding rule $L\&, L\vee, L\exists$ is invertible, so we can work as before with $\phi, \Pi \Rightarrow \forall\psi, \Phi'$.

Case $L\exists$:

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \psi x, \Pi' \Rightarrow \Phi}{\phi, \exists\psi, \Pi' \Rightarrow \Phi} L\exists}{\Gamma, \exists\psi, \Pi' \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \psi x, \Pi' \Rightarrow \Phi}{\Gamma, \psi x, \Pi' \Rightarrow \Delta, \Phi} Cut}{\Gamma, \exists\psi, \Pi' \Rightarrow \Delta, \Phi} L\exists$$

with x in Γ, Δ being renamed as usual, with one cut of rank $\leq (s, n, n' - 1)$.

Case $R\exists$:

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \frac{\phi, \Pi \Rightarrow \exists\psi, \psi t, \Phi'}{\phi, \Pi \Rightarrow \exists\psi, \Phi'} R\exists}{\Gamma, \Pi \Rightarrow \exists\psi, \Delta, \Phi'} Cut$$

is transformed to

$$\frac{\Gamma \Rightarrow \Delta, \phi \quad \phi, \Pi \Rightarrow \exists\psi, \psi t, \Phi'}{\Gamma, \Pi \Rightarrow \Delta, \exists\psi, \psi t, \Phi'} Cut}{\Gamma, \Pi \Rightarrow \exists\psi, \Delta, \Phi'} R\exists$$

with one cut of rank $\leq (s, n, n' - 1)$

Case 4: Each premise is obtained by a rule having ϕ as the principal formula:

Case $\phi = \alpha \supset \beta$:

$$\frac{\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \Delta, \alpha \supset \beta} R\supset \quad \frac{\alpha \supset \beta, \Pi \Rightarrow \alpha \quad \beta, \Pi \Rightarrow \Phi}{\alpha \supset \beta, \Pi \Rightarrow \Phi} L\supset}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \alpha \supset \beta \quad \alpha \supset \beta, \Pi \Rightarrow \alpha}{\Gamma, \Pi \Rightarrow \Delta, \alpha} Cut \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \beta} Cut \quad \beta, \Pi \Rightarrow \Phi}{\frac{\Gamma, \Gamma, \Pi, \Pi \Rightarrow \Delta, \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Contr} Cut$$

with three cuts of rank $\leq (s, n, n' - 1)$, $(s - 1, X, X)$ and $(s - 1, X, X)$.

Case $\phi = \alpha \& \beta$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \& \beta} R\& \quad \frac{\alpha, \beta, \Pi \Rightarrow \Phi}{\alpha \& \beta, \Pi \Rightarrow \Phi} L\&}{\Gamma, \Pi \Rightarrow \Delta, \Phi} Cut$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \alpha \quad \frac{\Gamma \Rightarrow \Delta, \beta \quad \alpha, \beta, \Pi \Rightarrow \Phi}{\Gamma, \alpha, \Pi \Rightarrow \Delta, \Phi} \text{Cut}}{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Phi} \text{Cut}}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Contr}$$

with two cuts of rank $\leq (s-1, X, X)$.

Case $\phi = \alpha \vee \beta$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} R\vee \quad \frac{\alpha, \Pi \Rightarrow \Phi \quad \beta, \Pi \Rightarrow \Phi}{\alpha \vee \beta, \Pi \Rightarrow \Phi} L\vee}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Cut}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \alpha, \beta \quad \alpha, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \beta, \Phi} \text{Cut} \quad \beta, \Pi \Rightarrow \Phi}{\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Phi, \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Contr}} \text{Cut}$$

with two cuts of rank $\leq (s-1, X, X)$.

Case $\phi = \forall \psi$:

$$\frac{\frac{\Gamma \Rightarrow \psi x}{\Gamma \Rightarrow \Delta, \forall \psi} R\forall \quad \frac{\forall \psi, \psi t, \Pi \Rightarrow \Phi}{\forall \psi, \Pi \Rightarrow \Phi} L\forall}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Cut}$$

is transformed to

$$\frac{\Gamma \Rightarrow \psi t \quad \frac{\Gamma \Rightarrow \Delta, \forall \psi \quad \forall \psi, \psi t, \Pi \Rightarrow \Phi}{\psi t, \Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Cut}}{\frac{\Gamma, \Gamma, \Pi \Rightarrow \Delta, \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Contr}} \text{Cut}$$

where $\Gamma \Rightarrow \psi t$ comes from $\Gamma \Rightarrow \psi x$ by substitution, since x cannot occur freely in Γ . The two cuts have rank $\leq (s, n, n'-1)$ and $(s-1, n-1, X)$.

Case $\phi = \exists \psi$:

$$\frac{\frac{\Gamma \Rightarrow \Delta, \exists \psi, \psi t}{\Gamma \Rightarrow \Delta, \exists \psi} R\exists \quad \frac{\psi x, \Pi \Rightarrow \Phi}{\exists \psi, \Pi \Rightarrow \Phi} L\exists}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Cut}$$

is transformed to

$$\frac{\frac{\Gamma \Rightarrow \Delta, \exists \psi, \psi t \quad \exists \psi, \Pi \Rightarrow \Phi}{\Gamma, \Pi \Rightarrow \Delta, \psi t, \Phi} \text{Cut} \quad \psi t, \Pi \Rightarrow \Phi}{\frac{\Gamma, \Pi, \Pi \Rightarrow \Delta, \Phi, \Phi}{\Gamma, \Pi \Rightarrow \Delta, \Phi} \text{Contr}} \text{Cut}$$

where $\psi t, \Pi \Rightarrow \Phi$ comes from $\psi x, \Pi \Rightarrow \Phi$ by substitution, since x cannot occur freely in $\Pi \Rightarrow \Phi$. The two cuts have rank $\leq (s, n-1, n')$ and $(s-1, X, n'-1)$.

□

7 Acknowledgements

We thank Albert Dragalin for an explanation of his proof, Sara Negri for collaboration on [3] and Max Dyckhoff for L^AT_EX efforts and expertise.

References

- [1] A. G. Dragalin. “Mathematical Intuitionism”, Translations of Mathematical Monographs, vol. 67, American Mathematical Society, Providence, Rhode Island, 1988.
- [2] R. Dyckhoff. *Contraction-free sequent calculi for intuitionistic logic*, Journal of Symbolic Logic, vol. 57, no. 3, pp. 795–807, 1992.
- [3] R. Dyckhoff, S. Negri. *Direct admissibility of structural rules for contraction-free sequent calculi*, in preparation, August 1997.
- [4] A. S. Troelstra, H. Schwichtenberg. “Basic Proof Theory”, Cambridge University Press, 1996.
- [5] A. S. Troelstra, H. Schwichtenberg. Corrections to “Basic Proof Theory”, “<http://turing.wins.uva.nl/~anne/Corr/bptcorr.ps>”, 1997.