Admissibility of Structural Rules for Extensions of Contraction-free Sequent Calculi

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Abstract

The contraction-free sequent calculus **G4** for intuitionistic logic is extended by rules following a general rule-scheme for nonlogical axioms. Admissibility of structural rules for these extensions is proved in a direct way by induction on derivations. This method permits the representation of various applied logics as complete, contraction- and cut-free sequent calculus systems with some restrictions on the nature of the derivations. As specific examples, intuitionistic theories of apartness and order and (Robinson-style) arithmetic are treated.

Keywords: applied sequent calculus, contraction-free, apartness, conservativity, cut-elimination

1 Introduction

Extension of a cut-free sequent calculus with new rules can easily destroy the cut-free nature of the calculus. In other words, if two calculi are equivalent in the sense of deriving the same sequents, the addition of some new rules to each calculus can generate non-equivalent calculi. In this paper we address a particular example where, we show, two equivalent calculi for intuitionistic propositional logic can be extended with rules representing non-logical axioms without the equivalence being lost.

Our interest in this topic began with the old question of the conservativity of apartness over equality; similar questions arise in the context of work such as [7, 8] on positive counterparts of classical notions of order. Addressing this question, the second author has shown how to extend Troelstra's single succedent version **G3ip** [13] of the Kleene-Dragalin sequent calculus with rules for apartness in such a way that *Cut* is admissible for the extension, thus answering the conservativity question affirmatively [4]. Similar results hold for the positive counterparts of order. The second author and Jan von Plato in [6] generalised part of this approach by giving a method for extending a sequent calculus with inference rules for nonlogical axioms; this works in classical logic for all kinds of axiom, and in intuitionistic logic for some kinds of axiom, including those considered in [4].

An earlier approach to the conservativity question, however, used as a basis not **G3ip** but the terminating "contraction-free" calculus **G4ip** (cf. [14, 2] and other references cited in the latter) for intuitionistic propositional logic, because of the extra restrictions on derivations therein compared to **G3ip**. However, it was not clear that the usual structural rules (e.g. *Cut*) admissible in **G4ip** were still admissible after the addition of new rules, for the new rules, like quantifier rules, can break the calculus' termination property, essential in the proof [2] of admissibility of the structural rules.

To address this problem we developed direct proofs [3] of admissibility of the structural rules for **G4ip** in the absence of nonlogical rules. These proofs used methods that extend to the extension of the calculus with rules for quantifiers.

The present paper shows how the same direct proofs may be extended to **G4ip**-based sequent calculi with rules for nonlogical axioms. A separate report [5], too long to include here, gives the details of the earlier proof of the conservativity result, now justified by the completeness argument of the present paper.

2 Background

We summarise first the approach of [3]. We considered there the sequent calculus **G4ip** for intuitionistic propositional logic, with no primitive structural rules; sequents have multiset antecedents and the left implication rule is split into the following four rules according to the form of the antecedent A of the principal formula $A \supset B$

$$\frac{P, B, \Gamma \Rightarrow E}{P, P \supset B, \Gamma \Rightarrow E}\, L0\supset \qquad \frac{C \supset (D \supset B), \Gamma \Rightarrow E}{(C \& D) \supset B, \Gamma \Rightarrow E}\, L\&\supset$$

$$\frac{C \supset B, D \supset B, \Gamma \Rightarrow E}{(C \vee D) \supset B, \Gamma \Rightarrow E}\, L\vee\supset \qquad \frac{C, D \supset B, \Gamma \Rightarrow D \quad B, \Gamma \Rightarrow E}{(C \supset D) \supset B, \Gamma \Rightarrow E}\, L\supset\supset$$

in which P is atomic (i.e. a propositional variable).

The calculus thus obtained has a major advantage over other systems; it is terminating, i.e. there is a bound on the depths of proofs, as a function of the weight of the end-sequent. Note that in **G3ip** the principal formula of the left implication rule has to be duplicated into the first premiss, to avoid loss of completeness, but that this allows proofs to be of unbounded depth.

Completeness of **G4ip** is shown by demonstration of the admissibility of the structural rules *Weakening*, *Contraction* and *Cut*. The first is straightforward. Next, one shows, by routine inductions on derivation height, the invertibility of almost all the primitive rules, the exceptions being $R\vee$ and $L\supset\supset$. To show admissibility of *Contraction* one uses these inversions, as in [1], to replace unwanted occurrences of the contracted formula by less complex formulas, thus appealing to an induction on the weight of such formulas. This technique fails for the cases where the inference above the contraction inference is not invertible (and where one of the contracted formulas is principal); so the only difficult case is rule $L\supset\supset$. This case is handled by Lemma 4.2 of [3], which shows (roughly) that the rule **is** invertible in the special case being considered, provided one can assume contraction-admissibility for less complex formulas. Admissibility of *Cut* is then routine.

This proof extends with minor changes to a multi-succedent calculus (already treated by other methods in [2]) and to the first-order calculus. In the latter case, one no longer has any bounds on the depths of proofs, but at least the propositional system is a subsystem of the first-order system, with a consequent advantage for implementation.

Second, we consider the work of Negri and von Plato [6]; this addresses the problem of the lack of cut elimination for sequent calculi extended with nonlogical axioms, and shows that cut-free extensions of **G3**-based classical and intuitionistic systems are obtained by adding appropriate *nonlogical rules* in place of nonlogical axioms. For example, an axiom of the form

$$P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$$

is replaced by a rule

$$\frac{P_1, \ldots, P_m, Q_1, \Gamma \Rightarrow \Delta \quad \ldots \quad P_1, \ldots, P_m, Q_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}\ \textit{rule-scheme}$$

with the principal formulas $P_1, \ldots, P_m$ copied into the premisses in order to ensure admissibility of contraction.

The axioms for apartness may accordingly be replaced by addition of the rules (the first with zero premisses)

$$\frac{}{a \neq a, \Gamma \Rightarrow \Delta}\ \textit{irref}$$

and

$$\frac{a \neq b, a \neq c, \Gamma \Rightarrow \Delta \qquad a \neq b, b \neq c, \Gamma \Rightarrow \Delta}{a \neq b, \Gamma \Rightarrow \Delta}\ \textit{split}$$

to the multi-succedent system **G3ipm** for intuitionistic propositional logic.

Admissibility of the standard structural rules, subject to a minor constraint (the *closure condition*, on the closure of the new set of rules under certain simplifications), is then shown in [6].

In the following, we consider both single-succedent and multi-succedent calculi; similar results hold whichever is used.

3 Extension with rules for apartness

The (free variable) theory of apartness is obtained as above by adding to **G3ip** the sequent calculus rules *irref* and *split*. The language is no longer based on proposition variables but on atoms of the form $a \neq b$, read as "a is *apart* from b." Equality can then be defined by $a = b =_{def} \neg(a \neq b)$. In the calculus **G3AP** so obtained, *Contraction* and *Cut* are admissible [4].

We can change the logical part of **G3AP** in favour of **G4ip** in order to obtain a calculus with better control on the structure of derivations, to be called **G4AP**. The constraints on the structure of derivations in this and in other **G4**-based calculi are given by the replacement of the left rule for implication by the four rules that analyze the logical structure of the antecedent, and especially by the use of the rule $L0\supset$; see page 5 of [5] for more discussion

The next step is to prove that in **G4AP** *Contraction* and *Cut* are admissible. To this end, since *Contraction* and *Cut* have been proved to be admissible in **G3AP**, it would be enough to prove the equivalence between **G3AP** and **G4AP** that extends the equivalence between **G3ip** and **G4ip**.

The "old" proof of equivalence between **G3ip** and **G4ip** (cf. [2]) relies on the property of rules of **G4ip** of being strictly monotonic (when read from premiss(es) to conclusion) with respect to the weight of sequents. This monotonicity is destroyed by adding a rule like *split*, where the premisses are of greater weight than the conclusion. As a consequence, the old proof of equivalence between the logical calculi cannot be extended to a proof of equivalence between **G3AP** and **G4AP**. Similar remarks apply to all other proofs of the equivalence that are referenced in [2]. This specific problem led to the demand for a direct syntactic proof of admissibility of *Cut* for **G4ip**.

We now show how the results in [3] extend to a proof of admissibility of *Cut* for **G4AP** and thus to equivalence between **G4AP** and **G3AP**.

We observe that the principal formulas of the rules added to **G4ip** to obtain **G4AP** are atomic. This implies that all the inversion lemmas of **G4ip** also hold for **G4AP**. For similar reasons, the rule W of (left) *Weakening* is admissible in **G4AP**.

LEMMA 3.1
The rule

$$\frac{\Gamma \Rightarrow D \quad B, \Gamma \Rightarrow E}{D \supset B, \Gamma \Rightarrow E}$$

is admissible in **G4AP**.

Proof: By induction on the height n of the derivation d of the first premiss. If $n = 0$, then the premiss is an axiom; if $\bot \in \Gamma$, then the conclusion is an axiom, and if D is an atom in Γ, then the conclusion follows by applying $L0\supset$ to the second premiss. If $a \neq a \in \Gamma$ then the conclusion also is an axiom. Now let $n > 0$. By the proof of Lemma 4.1 of [3], we only have to consider the case in which the last rule of d is *split*. We transform the derivation

$$\cfrac{\cfrac{a \neq b, a \neq c, \Gamma' \Rightarrow D \quad a \neq b, b \neq c, \Gamma' \Rightarrow D}{a \neq b, \Gamma' \Rightarrow D}\,split \qquad a \neq b, B, \Gamma' \Rightarrow E}{a \neq b, D \supset B, \Gamma' \Rightarrow E}$$

into the derivation

$$\cfrac{\cfrac{a \neq b, a \neq c, \Gamma' \Rightarrow D \quad \cfrac{a \neq b, B, \Gamma' \Rightarrow E}{a \neq b, a \neq c, B, \Gamma' \Rightarrow E}\,W}{a \neq b, a \neq c, D \supset B, \Gamma' \Rightarrow E}\,Ind \qquad \cfrac{a \neq b, b \neq c, \Gamma' \Rightarrow D \quad \cfrac{a \neq b, B, \Gamma' \Rightarrow E}{a \neq b, b \neq c, B, \Gamma' \Rightarrow E}\,W}{a \neq b, b \neq c, D \supset B, \Gamma' \Rightarrow E}\,Ind}{a \neq b, D \supset B, \Gamma' \Rightarrow E}\,split$$

□

The proof of Lemma 4.2 of [3] extends to the theory of apartness since only atomic formulas are principal in *split*, so we have:

LEMMA 3.2
The rule

$$\frac{(C \supset D) \supset B, \Gamma \Rightarrow E}{C, D \supset B, D \supset B, \Gamma \Rightarrow E}$$

is admissible in **G4AP**.

PROPOSITION 3.3
The *Contraction* rule

$$\frac{A, A, \Gamma \Rightarrow E}{A, \Gamma, \Rightarrow E} \; Contr$$

is admissible in **G4AP**.

Proof: The proof proceeds as in Proposition 5.1 of [3], and we only have to add the case in which A is principal and the last rule applied is *split*. In this case we apply the inductive hypothesis to the premisses of *split* and then obtain the conclusion by *split*. □

PROPOSITION 3.4
The rule

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow E}{A \supset B, \Gamma \Rightarrow E} \; L\supset$$

is admissible in **G4AP**.

Proof: Weaken the second premiss with $A \supset B$, use Lemma 3.1 and contract $A \supset B$. □

It follows that all the rules of **G3AP** are admissible in **G4AP**. The converse follows from admissibility of *Cut* in **G3AP**; so the two calculi are equivalent.

THEOREM 3.5
The *Cut* rule

$$\frac{\Gamma \Rightarrow A \quad A, \Gamma' \Rightarrow E}{\Gamma, \Gamma' \Rightarrow E} \; Cut$$

is admissible in **G4AP**.

Proof: We could use the equivalence between **G4AP** and **G3AP** and admissibility of *Cut* in the latter. We shall however give a direct proof. If $\Gamma \Rightarrow A$ or $A, \Gamma' \Rightarrow E$ is a logical axiom, then we proceed as in the proof [3] of admissibility of *Cut* for **G4ip**. If the left premiss is by *irref*, then also the conclusion is an axiom. If the right premiss is by *irref*, with $a \neq a \in \Gamma'$, then also the conclusion is an axiom. If instead $A \equiv a \neq a$ and the left premiss is not an axiom, since A is atomic, the last rule of the derivation of the left premiss has to be a left rule, including possibly *split*. So we apply *Cut* to the premisses of this rule and the right premiss, and then the rule. As for the case of A principal in both premisses, we observe that succedent atomic formulas cannot be principal in the calculus **G4AP**, so here nothing has to be added to the proof of admissibility of *Cut* done for **G4ip**. Note that we use the admissibility of *Weakening* and *Contraction* mentioned or proved above. □

4 Extension with rules for order

In [4] sequent calculi for positive theories of order, based on [7, 8], are obtained by adding to **G3ip** suitable sequent calculus rules. The same rules can be added to **G4ip**, obtaining calculi in which, as we shall see, the structural rules are admissible.

The language is based on atoms of the form $a \nleqslant b$, read as "*a exceeds b*." The usual notion of partial order can then be defined using $a \leqslant b =_{def} \neg(a \nleqslant b)$.

The sequent calculus **G4PPO** for the *positive theory of partial order* is obtained by adding to **G4ip** the rules

$$\frac{}{a \nleq a, \Gamma \Rightarrow C}\ irref \qquad \frac{a \nleq b, a \nleq c, \Gamma \Rightarrow C \quad a \nleq b, c \nleq b, \Gamma \Rightarrow C}{a \nleq b, \Gamma \Rightarrow C}\ split$$

The calculus **G4PLO** for *constructive linear order* is obtained by adding to **G4PPO** the rule

$$\frac{}{a \nleq b, b \nleq a, \Gamma \Rightarrow C}\ asym$$

The calculus **G4PLT** for *positive lattices* is obtained by using the infix term operators $\vee$ and $\wedge$ for join and meet, adding to **G4PPO** the appropriate (positive) rules:

$$\frac{}{a \nleq a{\vee}b, \Gamma \Rightarrow C}\ jnl \qquad \frac{}{b \nleq a{\vee}b, \Gamma \Rightarrow C}\ jnr$$

$$\frac{}{a{\wedge}b \nleq a, \Gamma \Rightarrow C}\ mtl \qquad \frac{}{a{\wedge}b \nleq b, \Gamma \Rightarrow C}\ mtr$$

$$\frac{a{\vee}b \nleq c, a \nleq c, \Gamma \Rightarrow C \qquad a{\vee}b \nleq c, b \nleq c, \Gamma \Rightarrow C}{a{\vee}b \nleq c, \Gamma \Rightarrow C}\ jnu$$

$$\frac{c \nleq a{\wedge}b, c \nleq a, \Gamma \Rightarrow C \qquad c \nleq a{\wedge}b, c \nleq b, \Gamma \Rightarrow C}{c \nleq a{\wedge}b, \Gamma \Rightarrow C}\ mtu$$

These reduce to rules for minimum and maximum in the case of linear order.

The calculus **G4PHA** for *positive Heyting algebras* is obtained by using the constant symbol 0 for the "bottom" element and the infix term operator $\rightarrow$ for Heyting arrow and by adding to **G4PLT** the following sequent calculus rules, corresponding to the axioms PHI and PHU given in [7]:

$$\frac{}{(a{\rightarrow}b){\wedge}a \nleq b, \Gamma \Rightarrow C}\ phi \qquad \frac{c \nleq a{\rightarrow}b, c{\wedge}a \nleq b, \Gamma \Rightarrow C}{c \nleq a{\rightarrow}b, \Gamma \Rightarrow C}\ phu$$

The zero-premiss rule giving the characteristic property of 0 is

$$\frac{}{0 \nleq a, \Gamma \Rightarrow C}\ phb$$

The proofs of admissibility of *Contraction* and *Cut* for the theories of positive order based on **G4ip** follow the pattern given in [4] for theories based on **G3ip** and specialized in the previous section for theory of apartness based on **G4ip**; therefore we will not enter into the details here.

THEOREM 4.1
The rules of *Contraction* and *Cut* are admissible in **G4PPO**, **G4PLO**, **G4PLT**, and **G4PHA**.

5 Extension with rules for nonlogical axioms

The proof of admissibility of structural rules for **G4ip** extended with rules following the rule-scheme can be obtained along the lines shown in the previous sections for apartness and positive order. We limit ourselves to the following remarks:

1. Only atomic formulas occur as active and principal formulas in nonlogical rules following the rule-scheme, thus the invertibility properties of the logical rules of **G4** extended with nonlogical rules are not affected.
2. The repetition of the principal formulas in the premisses of the rule-scheme makes possible the inductive step in the proof of admissibility of *Contraction* when only one of the contraction formulas is principal in the nonlogical rule. The case of both contraction formulas principal in the nonlogical rule is handled using the *closure condition*, as explained in [6].

The above two features of the rule-scheme ensure its modularity when a direct proof, by induction on derivation height and formula weight, of admissibility of structural rules is required. Arguments based on induction on sequent weight, on the other hand, do not appear to extend to calculi augmented with the rule-scheme.

As an application of the method, we can answer a question raised by the referee. One can easily obtain a contraction- and cut-free calculus for RHA, Robinson [Heyting] Arithmetic, by converting the axioms for equality, successor, predecessor, and arithmetic operations into rules following the rule-scheme; first, a sequent system for equality is obtained by adding to the logical calculus the rule of reflexivity and the atomic replacement scheme

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{refl} \qquad \frac{a = b, P(a), P(b), \Gamma \Rightarrow \Delta}{a = b, P(a), \Gamma \Rightarrow \Delta}\ \mathit{repl}$$

As shown in [6] the above two rules, when added to classical or intuitionistic **G3**, give a complete contraction- and cut-free system for equality. The same holds if the above rules are added to a **G4**-system. The arithmetic axioms

1. $a = pd(s(a))$
2. $\neg s(a) = 0$
3. $a + 0 = a$
4. $a + s(b) = s(a + b)$
5. $a \cdot 0 = 0$
6. $a \cdot s(b) = a \cdot b + a$

are translated into quantifier-free nonlogical rules, in compliance with the above guidelines, as follows:

$$\frac{a = pd(s(a)), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{}{s(a) = 0, \Gamma \Rightarrow \Delta}$$

$$\frac{a + 0 = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{a + s(b) = s(a + b), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

$$\frac{a \cdot 0 = 0, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \frac{a \cdot s(b) = a \cdot b + a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Finally, Robinson's axiom

$$\forall x(x = 0 \vee x = s(pd(x)))$$

becomes the nonlogical rule

$$\frac{a = 0, \Gamma \Rightarrow \Delta \quad a = s(pd(a)), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

One thus obtains calculi **G3RHA** and **G4RHA**; the completeness of the former follows by the methods of [6]. By the methods sketched above, one has also

THEOREM 5.1
The rules of *Contraction* and *Cut* are admissible in **G4RHA**.

6 Conclusion and related work

We have shown that the two modifications, addition [6] of nonlogical rules (in a certain standard form) and the replacement [2] of the left implication rule by related rules that constrain the proofs rather tightly, may be made to the standard **G3ip** calculus *independently*, i.e., without the addition of one affecting the completeness proof of the other. Elsewhere [5] we illustrate the power of such extended calculi by simplifying the proof of the major conservativity result of [4].

Finally, following a referee's suggestion, we mention the existence of papers from the Leningrad school in the sixties and seventies that contain various suggestions for converting nonlogical axioms to nonlogical rules, such as [9, 10, 11, 12]. These papers do not cover the extension of constructive systems, which are, together with a careful handling of contraction, the central issue both of [6] and of the present work.

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