Admissibility of structural rules for contraction-free systems of intuitionistic logic

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Abstract

We give a direct proof of admissibility of cut and contraction for the contraction-free sequent calculus **G4ip** for intuitionistic propositional logic and for a corresponding multi-succedent calculus; this proof extends easily in the presence of quantifiers, in contrast to other, indirect, proofs, i.e. those which use induction on sequent weight or appeal to admissibility of rules in other calculi.

1 Introduction

In this work we present a direct proof of admissibility of cut and other structural rules for a certain sequent calculus for intuitionistic logic. This calculus is called **G4i** in [26], of which we henceforth follow the notation (except for $A \supset B$ in place of $A \to B$). Several papers ([6], [11], [12], [13], [15], [28], [29]) and the book [26] have shown the admissibility of contraction and cut for **G4ip**, the propositional part of **G4i**, or for its variants. These calculi have (in contrast to the calculi **G1ip**, **G2ip** and **G3ip**) the useful feature that root-first proof search terminates without any loop-detection; this is exploited in various implementations ([2], [8], [5], [13], [20], [9], [16], [3], [24] and [25]) and in Pitts' influential work [22] on second-order quantification.

All these proofs of admissibility use inductions on formula weights and on derivation heights; these are unproblematic. But they also use the corresponding results for **G3ip** and an induction on sequent weight (based on the weights of formulae in the sequent); their extension, for extensions of **G4ip**, is therefore problematic, unless the extensions use only rules in which the weight of each premise is less than that of the conclusion. (There are also non-constructive, model-theoretic approaches in [21], [1] and [17]: however, we prefer constructive reasoning.)

We present here a direct proof (along the lines, using inversion lemmas, exploited by Dragalin [4]) of the admissibility of contraction and cut for **G4ip**. We show how it can be extended when one adds to **G4ip** the rules (Section 8) for first-order syntax (as in [10]). The proof adapts with little difficulty to the multi-succedent calculus **G4ip′** (Section 7).

Our new proof is *direct*, in that it makes no use of similar metatheorems for **G3ip** and no use of induction on sequent weight. It is also routine; all details can easily be filled in by any reader familiar with the technique in [4]. In [6] the first author wrote that "a direct proof of cut-elimination ... seems difficult". Now that we have it, we regard the indirect proofs as the difficult ones. The difficulty was in the proof of admissibility of contraction; this is overcome by means of two lemmas, 4.1 and 4.2.

2 Background

Zero-order and first-order formulae A, B, C, D, E are built up as usual: but P and Q range over atomic formulae and $\bot$ is not an atomic formula. In zero-order examples $p, q, \ldots$ are proposition variables, i.e. atomic formulae. Γ, Δ and Θ range over multisets of formulae. *Judgments* are of the form $\Gamma \Rightarrow A$. The primitive rules of the calculus **G4ip** are

$$\frac{}{\Gamma, P \Rightarrow P}\ Axiom \qquad \frac{}{\Gamma, \bot \Rightarrow E}\ L\bot$$

$$\frac{\Gamma, A, B \Rightarrow E}{\Gamma, A\&B \Rightarrow E}\ L\& \qquad \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A\&B}\ R\&$$

$$\frac{\Gamma, A \Rightarrow E \quad \Gamma, B \Rightarrow E}{\Gamma, A \vee B \Rightarrow E}\ L\vee \qquad \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_0 \vee A_1}\ R\vee$$

$$\frac{\Gamma, P, B \Rightarrow E}{\Gamma, P, P \supset B \Rightarrow E}\ L0\supset \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}\ R\supset$$

$$\frac{\Gamma, C \supset (D \supset B) \Rightarrow E}{\Gamma, (C\&D) \supset B \Rightarrow E}\ L\&\supset \qquad \frac{\Gamma, C \supset B, D \supset B \Rightarrow E}{\Gamma, (C \vee D) \supset B \Rightarrow E}\ L\vee\supset$$

$$\frac{\Gamma, C, D \supset B \Rightarrow D \quad \Gamma, B \Rightarrow E}{\Gamma, (C \supset D) \supset B \Rightarrow E}\ L\supset\supset$$

Note that we use a slight variant of the $L\supset\supset$ rule used in [6] and [26], and that in axioms $\Gamma, P \Rightarrow P$ and rules $L0\supset$ the formula P is atomic. Loosely, we refer to sequents $\Gamma, \bot \Rightarrow E$ as "axioms". *Derivations* are the labelled trees whose leaves are axioms and whose other nodes match rules: each node contains a judgment and each internal node is labelled by the name of the rule or of the lemma where the rule is shown to be admissible. We use the label *Ind* where appealing to an induction hypothesis.

The *weight* $w(A)$ of a formula A is defined as follows:

$w(\bot) = 0$,

$w(P) = 1$ for any atomic formula P,

$w(A \supset B) = 1 + w(A) + w(B)$,

$w(A\&B) = 2 + w(A) + w(B)$,

$w(A \vee B) = 3 + w(A) + w(B)$.

"Lighter" is synonymous with "of lower weight". We use induction on formula weight rather than on formula size or sequent weight. The *height* of a derivation (using primitive rules) is just its height as a tree; so a tree with one node has height 0. "Height" is undefined for derivations using the non-primitive rules.

Let G be a logical calculus. We recall that a (schematic) rule

$$\frac{S}{S'}$$

in G is *admissible* in G iff for every derivation of an instance of S there is one, of the corresponding instance, of S'. (Similarly for multi-premise rules.)

Definition 2.1 *The rule*

$$\frac{S}{S'}$$

is strongly admissible *iff for every n and every derivation of height n of an instance of S there is a derivation of height $\leq n$ of the corresponding instance of S'.*

We don't use anywhere the strong admissibility of rules but include some results about strong admissibility for future reference.

Lemma 2.2 *The* Weakening *rule*

$$\frac{\Gamma \Rightarrow A}{\Gamma, \Gamma' \Rightarrow A}\, W$$

is strongly admissible.

Proof: Routine induction on the height of the derivation of the premise. □

3 Basic lemmas

We give here some routine lemmas similar to those established in [4] prior to the proof of admissibility of *Contraction* and not really specific to the "contraction-free" approach of **G4ip**. Our inversion lemmas can be stated as (strong) admissibility of certain rules, converses of the primitive inference rules.

Lemma 3.1 *The following rules are strongly admissible in* **G4ip***:*

1. $\dfrac{\Gamma, A\&B \Rightarrow E}{\Gamma, A, B \Rightarrow E}$;
2. $\dfrac{\Gamma \Rightarrow A\&B}{\Gamma \Rightarrow A}$; $\dfrac{\Gamma \Rightarrow A\&B}{\Gamma \Rightarrow B}$;
3. $\dfrac{\Gamma, A \vee B \Rightarrow E}{\Gamma, A \Rightarrow E}$; $\dfrac{\Gamma, A \vee B \Rightarrow E}{\Gamma, B \Rightarrow E}$;
4. $\dfrac{\Gamma, (C\&D)\supset B \Rightarrow E}{\Gamma, C\supset(D\supset B) \Rightarrow E}$;
5. $\dfrac{\Gamma, (C \vee D)\supset B \Rightarrow E}{\Gamma, C\supset B, D\supset B \Rightarrow E}$;
6. $\dfrac{\Gamma, P\supset B \Rightarrow E}{\Gamma, B \Rightarrow E}$;
7. $\dfrac{\Gamma, (C\supset D)\supset B \Rightarrow E}{\Gamma, B \Rightarrow E}$;
8. $\dfrac{\Gamma \Rightarrow A\supset B}{\Gamma, A \Rightarrow B}$.

Proof: By induction on the height n of the derivation d of the premise of each item. If $n = 0$ the premise is an axiom and so the conclusion is an axiom. (Sequents of the form $\Gamma, E \Rightarrow E$ are axioms iff $\bot \in \Gamma$ or E is an atom.) If $n > 0$ we distinguish two cases, according to whether the main formula (the one in the premise that does not appear in the conclusion) of the inference is principal or not in the last step of d. If it is principal, then a premise of the last inference gives the conclusion; otherwise, one applies the inductive hypothesis to the premise(s) and then uses the rule again. □

Use of one of these rules will be indicated by *Inv* in derivation trees. In fact, the rules of **G4ip** are all invertible except for $R\vee$ and $L\supset\supset$; the latter is (partially) invertible w.r.t. the second premise, as seen in item 7 of the lemma.

Lemma 3.2 *Judgments of the following form*

1. $\Gamma, A \Rightarrow A$ *(generalized axiom)*
2. $\Gamma, A, A\supset B \Rightarrow B$ *(modus ponens)*

are derivable in **G4ip***.*

Proof: 1. By induction on $w(A)$. If A is $\bot$ or an atomic formula, then $\Gamma, A \Rightarrow A$ is an axiom. If $A = B\&C$, we have the derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma, B, C \Rightarrow B}\,Ind \quad \dfrac{}{\Gamma, B, C \Rightarrow C}\,Ind}{\Gamma, B, C \Rightarrow B\&C}\,R\&}{\Gamma, B\&C \Rightarrow B\&C}\,L\&$$

If $A = B \vee C$ we have the derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma, B \Rightarrow B}\,Ind}{\Gamma, B \Rightarrow B \vee C}\,R\vee_1 \quad \dfrac{\dfrac{}{\Gamma, C \Rightarrow C}\,Ind}{\Gamma, C \Rightarrow B \vee C}\,R\vee_2}{\Gamma, B \vee C \Rightarrow B \vee C}\,L\vee$$

If the outermost connective of A is an implication, we cannot prove the claim without relying on *2*, the proof of which, in turn, relies on *1*. This can be carried out by a simultaneous induction, but an alternative is to analyze the structure of the antecedent, as follows: If $A = P\supset B$ we have the derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma, P, B \Rightarrow B}\,Ind}{\Gamma, P, P\supset B \Rightarrow B}\,L0\supset}{\Gamma, P\supset B \Rightarrow P\supset B}\,R\supset$$

For $A = (C\&D)\supset B$ we have the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, C\supset(D\supset B) \Rightarrow C\supset(D\supset B)}\,Ind}{\Gamma, C\supset(D\supset B), C \Rightarrow D\supset B}\,3.1, 8}{\Gamma, C\supset(D\supset B), C, D \Rightarrow B}\,3.1, 8}{\Gamma, (C\&D)\supset B, C, D \Rightarrow B}\,L\&\supset}{\Gamma, (C\&D)\supset B, C\&D \Rightarrow B}\,L\&}{\Gamma, (C\&D)\supset B \Rightarrow (C\&D)\supset B}\,R\supset$$

where induction applies since $w(C\supset(D\supset B)) < w((C\&D)\supset B)$. For $A = (C\vee D)\supset B$, consider the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, C\supset B, D\supset B \Rightarrow C\supset B}\,Ind}{\Gamma, C\supset B, D\supset B, C \Rightarrow B}\,3.1,8 \qquad \dfrac{\dfrac{}{\Gamma, C\supset B, D\supset B \Rightarrow D\supset B}\,Ind}{\Gamma, C\supset B, D\supset B, D \Rightarrow B}\,3.1,8}{\dfrac{\Gamma, C\supset B, D\supset B, C\vee D \Rightarrow B}{\Gamma, (C\vee D)\supset B, C\vee D \Rightarrow B}\,L\vee\supset}\,L\vee}{\Gamma, (C\vee D)\supset B \Rightarrow (C\vee D)\supset B}\,R\supset$$

Finally, for $A = (C\supset D)\supset B$, the judgment is derived as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, D\supset B, C\supset D \Rightarrow C\supset D}\,Ind}{\Gamma, D\supset B, C\supset D, C \Rightarrow D}\,3.1,8 \qquad \dfrac{}{\Gamma, B, C\supset D \Rightarrow B}\,Ind}{\Gamma, (C\supset D)\supset B, C\supset D \Rightarrow B}\,L\supset\supset}{\Gamma, (C\supset D)\supset B \Rightarrow (C\supset D)\supset B}\,R\supset$$

2. By 1, the judgment $\Gamma, A\supset B \Rightarrow A\supset B$ is derivable, and the conclusion follows by invertibility of $R\supset$. □

4 Other key lemmas

We proceed by showing that a weak version of the rule $L\supset$ of **G3ip** is admissible in **G4ip**. The proof is similar to that of Lemma 4 in [12] of admissibility of the stronger rule $L\supset$, with $D\supset B$ in the antecedent of the first premise; as we mention in Section 9, that proof only works if D is atomic, because of an unconsidered case where $D\supset B$ is principal. By leaving the formula $D\supset B$ out of the premise we get a result weaker (in that the premise is stronger) but also stronger (D is unrestricted to being atomic).

Lemma 4.1 *The rule*

$$\dfrac{\Gamma \Rightarrow D \quad \Gamma, B \Rightarrow E}{\Gamma, D\supset B \Rightarrow E}$$

is admissible in **G4ip**.

Proof: By induction on the height n of the derivation d of the first premise. If $n = 0$, then the premise is an axiom: if $\bot \in \Gamma$, then the conclusion is an axiom, and if D is an atom, and $D \in \Gamma$, then the conclusion follows by applying $L0\supset$ to the second premise. Now let $n > 0$ and argue by cases.

1. If the last inference of d is by an invertible left rule, apply the corresponding inversion lemma to the right premise, then use the inductive hypothesis and the rule.

2. If the last inference of d is by $R\&$, with $D = D_1\&D_2$, apply the inductive hypothesis to the second premise and obtain $\Gamma, D_2\supset B \Rightarrow E$. Again by the inductive hypothesis, using the first premise, we get $\Gamma, D_1\supset(D_2\supset B) \Rightarrow E$ and the conclusion follows by $L\&\supset$.

3. If the last inference of d is by $R\vee$, use the inductive hypothesis, admissibility of weakening and $L\vee\supset$.

4. If the last inference of d is by $R\supset$, with premise $\Gamma, D_1 \Rightarrow D_2$, by admissibility of weakening we obtain $\Gamma, D_2 \supset B, D_1 \Rightarrow D_2$; by $L\supset\supset$ the conclusion $\Gamma, (D_1 \supset D_2) \supset B \Rightarrow E$ follows.

5. Finally, suppose that the last inference of d is by a non-invertible left rule, that is, $L\supset\supset$ with $\Gamma = \Gamma', (F \supset G) \supset H$ and premises $\Gamma', G \supset H, F \Rightarrow G$ and $\Gamma', H \Rightarrow D$. Thus we obtain

$$\dfrac{\dfrac{\Gamma', G \supset H, F \Rightarrow G}{\Gamma', G \supset H, F, D \supset B \Rightarrow G}\,W \qquad \dfrac{\Gamma', H \Rightarrow D \quad \dfrac{\Gamma', (F \supset G) \supset H, B \Rightarrow E}{\Gamma', H, B \Rightarrow E}\,Inv}{\Gamma', H, D \supset B \Rightarrow E}\,Ind}{\Gamma', (F \supset G) \supset H, D \supset B \Rightarrow E}\,L\supset\supset$$

□

The following lemma constitutes the essential step in the proof of admissibility of contraction.

Lemma 4.2 *The rule*

$$\frac{\Gamma, (C \supset D) \supset B \Rightarrow E}{\Gamma, C, D \supset B, D \supset B \Rightarrow E}$$

is admissible in **G4ip**.

Proof: By induction on the height n of the derivation of the premise. For $n = 0$, the premise is an axiom and the conclusion is an axiom. If the last inference is by a right rule or by a left rule with $(C \supset D) \supset B$ non-principal, the induction is straightforward. If the last inference is by $L\supset\supset$ with $(C \supset D) \supset B$ principal, the premises are $\Gamma, C, D \supset B \Rightarrow D$ and $\Gamma, B \Rightarrow E$; we now construct

$$\dfrac{\Gamma, C, D \supset B \Rightarrow D \qquad \dfrac{\Gamma, B \Rightarrow E}{\Gamma, C, D \supset B, B \Rightarrow E}\,W}{\Gamma, C, D \supset B, D \supset B \Rightarrow E}\,Lemma\ 4.1$$

□

The lemma is surprising because the repetition of $D \supset B$ in the conclusion might appear to make contraction-elimination harder rather than easier. However, the key point is that $D \supset B$ is a lighter formula than $(C \supset D) \supset B$.

5 Admissibility of Contraction

Proposition 5.1 *The* Contraction *rule*

$$\frac{\Gamma, A, A \Rightarrow E}{\Gamma, A \Rightarrow E}\,Contr$$

is admissible in **G4ip**.

Proof: By induction on the weight of the contracted formula A, and, for each weight, on the height n of the derivation d of the premise.

For A atomic or $\bot$, if $n = 0$, then the premise and the conclusion are both axioms. If $n > 0$, then A is not principal; the conclusion follows by applying induction to the premise(s) and then using the same rule as in the last inference of d.

For compound A, if $n = 0$, then $\bot \in \Gamma$ or $E \in \Gamma$ and the conclusion follows. If $n > 0$ and A is not principal, apply the inductive hypothesis to the premises and then use the rule again. For principal A, if A is $C\&D$, $C \vee D$, $P \supset B$, $(C\&D)\supset B$ or $(C \vee D)\supset B$ we proceed in a uniform way by using an inversion lemma (3.1), contraction on lighter formulae, and then the rule again. The handling of $A = (C\supset D)\supset B$ requires the use of Lemma 4.2: in this case the derivation ends with

$$\frac{\Gamma, D\supset B, (C\supset D)\supset B, C \Rightarrow D \quad \Gamma, (C\supset D)\supset B, B \Rightarrow E}{\Gamma, (C\supset D)\supset B, (C\supset D)\supset B \Rightarrow E}$$

and by the lemma the left premise gives $\Gamma, D\supset B, C, D\supset B, D\supset B, C \Rightarrow D$; thus by applying three contractions, on the lighter formulae C and $D\supset B$, we obtain $\Gamma, D\supset B, C \Rightarrow D$. The right premise gives, by an inversion lemma (3.1) and contraction on B, the sequent $\Gamma, B \Rightarrow E$. By $L\supset\supset$ the conclusion $\Gamma, A \Rightarrow E$ follows. □

Strictly speaking, uses of the *Contr* rule in the above proof are illegitimate, since the proof is for the calculus in which *Contr* is not primitive. We should interpret the above proposition just to mean that any **G4ip** derivation of the premise can be transformed to a **G4ip** derivation of the conclusion: the contractions apparently used in the results of transformation are smaller and thus can in turn be transformed, beginning if necessary with any topmost one. Similar remarks apply later to the proof of admissibility of *Cut*.

As a consequence of admissibility of contraction for **G4ip**, we obtain a direct proof of admissibility in **G4ip** of the rule $L\supset$ from **G3ip**:

Proposition 5.2 *The rule*

$$\frac{\Gamma, A\supset B \Rightarrow A \quad \Gamma, B \Rightarrow E}{\Gamma, A\supset B \Rightarrow E}\, L\supset$$

is admissible in **G4ip**.

Proof: Weaken the second premise with $A\supset B$, use Lemma 4.1 and contract $A\supset B$. □

It follows that all the rules of **G3ip** are admissible in **G4ip**. The converse is easy (since *Cut* is admissible in **G3ip**); so the two calculi are equivalent. As a consequence *Cut* is indirectly proved to be admissible in **G4ip**, but we will argue for admissibility of *Cut* directly.

In Lemma 3.1 partial invertibility of the rule $L\supset$ has been proved (as strong admissibility) only for antecedent A of the form P or $C\supset D$. As a consequence of admissibility of contraction, we are now able to prove the following for arbitrary antecedent A:

Proposition 5.3 *The rule*

$$\frac{\Gamma, A\supset B \Rightarrow E}{\Gamma, B \Rightarrow E}$$

is admissible in **G4ip**.

Proof: By induction on the weight of A with subinduction on the height n of the derivation d of $\Gamma, A\supset B \Rightarrow E$. For any A, if the premise is an axiom, also the conclusion is an axiom, so we suppose $n > 0$. If A is atomic, the conclusion follows by Lemma 3.1. If A is $\bot$, then $\bot \supset B$ is not principal in the last rule of d, so the conclusion follows by applying induction to the premise(s) of the rule and the rule again.

If A is compound and $A \supset B$ is not principal in the last rule of d, we argue as above. If $A \supset B$ is principal, the only cases yet to be considered are those with $A = C \& D$ and $A = C \vee D$. In the first case the premise of the last rule of D is $\Gamma, C \supset (D \supset B) \Rightarrow E$. By induction we get a derivation of $\Gamma, D \supset B \Rightarrow E$, and by induction again, since D is lighter than A, we obtain the conclusion. In the second case the premise is $\Gamma, C \supset B, D \supset B \Rightarrow E$, and we argue in a similar way to obtain a derivation of $\Gamma, B, B \Rightarrow E$. The conclusion then follows by admissibility of contraction. □

This proof does not extend to the quantifier case in Section 8; so, rather than using this lemma in the next section we argue directly, using induction and cut on lighter formulae.

6 Admissibility of Cut

Our proof is based on that in [4]; see also [7] for details.

Theorem 6.1 *The* Cut *rule*

$$\frac{\Gamma \Rightarrow A \quad \Gamma', A \Rightarrow E}{\Gamma, \Gamma' \Rightarrow E}\, Cut$$

is admissible in **G4ip**.

Proof: By induction on the weight of A, with a subsidiary induction on the sum of the heights of the derivations of $\Gamma \Rightarrow A$ and of $\Gamma', A \Rightarrow E$.

There are four cases:

1. $\Gamma \Rightarrow A$ or $\Gamma', A \Rightarrow E$ is an axiom;

2. Neither premise is an axiom and A is not principal in the left premise;

3. Neither premise is an axiom and A is principal in the left premise but not in the right premise;

4. A is principal in both premises.

The first and second cases are dealt with as in [4]; so is the third case for all rules, except when the derivation of the right premise ends with $L0\supset$; and so is the fourth case for $A = B \& C$ and $A = B \vee C$. In the third case, we can permute (A being principal in the left premise, it cannot be atomic)

$$\frac{\Gamma \Rightarrow A \quad \dfrac{A, P, B, \Gamma'' \Rightarrow E}{A, P, P \supset B, \Gamma'' \Rightarrow E}\, L0\supset}{\Gamma, P, P \supset B, \Gamma'' \Rightarrow E}\, Cut$$

to

$$\frac{\dfrac{\Gamma \Rightarrow A \quad A, P, B, \Gamma'' \Rightarrow E}{\Gamma, \Gamma'', P, B \Rightarrow E}\, Cut}{\Gamma, \Gamma'', P, P \supset B \Rightarrow E}\, L0\supset$$

The subcase of implication in case 4 splits into four sub-sub-cases:

1. $A = P \supset B$: The derivation ends as follows, where $\Gamma' = \Gamma'', P$:

$$\dfrac{\dfrac{\Gamma, P \Rightarrow B}{\Gamma \Rightarrow P\supset B}\,R\supset \qquad \dfrac{\Gamma'', P, B \Rightarrow E}{\Gamma', P\supset B \Rightarrow E}\,L0\supset}{\Gamma, \Gamma' \Rightarrow E}\,Cut$$

. We transform this into

$$\dfrac{\dfrac{\Gamma, P \Rightarrow B \quad \Gamma'', P, B \Rightarrow E}{\Gamma, \Gamma'', P, P \Rightarrow E}\,Cut}{\Gamma, \Gamma' \Rightarrow E}\,Contr$$

where the cut formula B is lighter than $P\supset B$.

2. $A = (C\&D)\supset B$: The derivation ends as follows:

$$\dfrac{\dfrac{\Gamma, C\&D \Rightarrow B}{\Gamma \Rightarrow (C\&D)\supset B}\,R\supset \qquad \dfrac{\Gamma', C\supset(D\supset B) \Rightarrow E}{\Gamma', (C\&D)\supset B \Rightarrow E}\,L\&\supset}{\Gamma, \Gamma' \Rightarrow E}\,Cut$$

We transform this into

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, C\&D \Rightarrow B}{\Gamma, C, D \Rightarrow B}\,Inv}{\Gamma, C \Rightarrow D\supset B}\,R\supset}{\Gamma \Rightarrow C\supset(D\supset B)}\,R\supset \qquad \Gamma', C\supset(D\supset B) \Rightarrow E}{\Gamma, \Gamma' \Rightarrow E}\,Cut$$

where the cut formula $C\supset(D\supset B)$ is lighter than $(C\&D)\supset B$.

3. $A = (C \vee D)\supset B$: Similar to the previous cases, using the inversion lemma for $L\vee$, cut (twice) on lighter formulae and contractions on Γ.

4. $A = (C\supset D)\supset B$: The derivation ends with

$$\dfrac{\dfrac{\Gamma, C\supset D \Rightarrow B}{\Gamma \Rightarrow (C\supset D)\supset B}\,R\supset \qquad \dfrac{\Gamma', D\supset B, C \Rightarrow D \quad \Gamma', B \Rightarrow E}{\Gamma', (C\supset D)\supset B \Rightarrow E}\,L\supset\supset}{\Gamma, \Gamma' \Rightarrow E}\,Cut$$

and is transformed into the following derivation with four cuts on the lighter formulae $D\supset B$, $C\supset D$ (twice) and B:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D, C \Rightarrow D}\,3.2}{D \Rightarrow C\supset D} \qquad \Gamma, C\supset D \Rightarrow B}{\Gamma, D \Rightarrow B}\,Cut}{\Gamma \Rightarrow D\supset B}\,R\supset \qquad \Gamma', D\supset B, C \Rightarrow D}{\Gamma, \Gamma', C \Rightarrow D}\,Cut}{\Gamma, \Gamma' \Rightarrow C\supset D}\,R\supset \qquad \Gamma, C\supset D \Rightarrow B}{\Gamma, \Gamma, \Gamma' \Rightarrow B}\,Cut \qquad \Gamma', B \Rightarrow E}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow E}\,Cut}{\Gamma, \Gamma' \Rightarrow E}\,Contr$$

.

□

We remark that in order to deal with the last case in the above proof we could have used Proposition 5.3 in order to obtain $\Gamma, D \Rightarrow B$ from $\Gamma, C \supset D \Rightarrow B$. But, as noted above, the proof of that proposition does not extend to the quantifier case. We could also replace the earlier uses of *Inv* by further cuts.

Corollary 6.2 *All instances of the Cut rule in a derivation in* **G4ip+Cut** *are eliminable.*

Proof: As usual, by induction on the number of instances, selecting for elimination any topmost cut instance. □

Of course, this argument is already needed in the proof of the theorem.

7 Extension with multiple succedents

We consider here the modifications needed for a multi-succedent calculus **G4ip′**, along the lines of the calculus GHPC of [4]. The primitive rules are modified to allow an arbitrary multiset of formulae in the succedent. Sequents $\Gamma \Rightarrow \Delta$ in which Γ and Δ have an atomic formula in common are axioms, as are those with $\bot \in \Gamma$. The two $R\vee$ rules merge into the single rule

$$\frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ R\vee$$

and the first premise of the $L\supset\supset$ rule has a single succedent (following [4] rather than 3.4.9D of [26]). Here are the rules:

$$\frac{}{\Gamma, P \Rightarrow P, \Delta}\ Axiom \qquad \frac{}{\Gamma, \bot \Rightarrow \Delta}\ L\bot$$

$$\frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A\&B \Rightarrow \Delta}\ L\& \qquad \frac{\Gamma \Rightarrow A, \Delta \quad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A\&B, \Delta}\ R\&$$

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}\ R\vee$$

$$\frac{\Gamma, P, B \Rightarrow \Delta}{\Gamma, P, P\supset B \Rightarrow \Delta}\ L0\supset \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A\supset B, \Delta}\ R\supset$$

$$\frac{\Gamma, C\supset(D\supset B) \Rightarrow \Delta}{\Gamma, (C\&D)\supset B \Rightarrow \Delta}\ L\&\supset \qquad \frac{\Gamma, C\supset B, D\supset B \Rightarrow \Delta}{\Gamma, (C \vee D)\supset B \Rightarrow \Delta}\ L\vee\supset$$

$$\frac{\Gamma, C, D\supset B \Rightarrow D \quad \Gamma, B \Rightarrow \Delta}{\Gamma, (C\supset D)\supset B \Rightarrow \Delta}\ L\supset\supset$$

The basic lemmas of Section 3 are still true, *mutatis mutandis*, except that $R\supset$ is no longer invertible. There are also new inversion lemmas for the rules $R\&$ and $R\vee$, needed because they have no copy of the principal formula in the premise(s). We also have:

Lemma 7.1 *Judgments of the following form*

1. $\Gamma, A \Rightarrow A, \Delta$ *(generalized axiom)*
2. $\Gamma, A, A\supset B \Rightarrow B, \Delta$ *(modus ponens)*

are derivable in **G4ip**′.

Proof: Roughly as before. Note that in the proof we need to use the invertibility of $R\supset$ only in the case where Δ is empty. □

Proposition 7.2 *The* Contraction-Right *rule*

$$\frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ \mathit{Contr\text{-}R}$$

is admissible in **G4ip**′.

Proof: Consider first the case where $A = A_1 \supset A_2$ is introduced by $R\supset$, with premise $\Gamma, A_1 \Rightarrow A_2$; from this we get $\Gamma \Rightarrow A, \Delta$ by $R\supset$. The other cases use induction and inversion lemmas as before. □

Lemma 7.3 *The rule*

$$\frac{\Gamma \Rightarrow D, \Theta \quad \Gamma, B \Rightarrow \Delta}{\Gamma, D\supset B \Rightarrow \Theta, \Delta}$$

is admissible in **G4ip**′.

Proof: By induction first on the weight of D and then on the height n of the derivation d of the first premise. Let d' be the derivation of the second premise.

For $n = 0$, the first premise is an axiom: if $\bot \in \Gamma$ or if Γ and Θ have an atom in common, then the conclusion is an axiom; and if D is an atom in Γ, use $L0\supset$ on d' and weaken with Θ.

For $n > 0$, argue by cases:

1. If d ends with an invertible left rule $L*$, we apply the inversion lemma to the conclusion of d', use the inductive hypothesis and then $L*$.

2. If d ends with $R\&$, there are two cases:

   (a) $D = D_1 \& D_2$ is principal, with premises $\Gamma \Rightarrow D_1, \Theta$ and $\Gamma \Rightarrow D_2, \Theta$: apply the inductive hypothesis to $\Gamma \Rightarrow D_2, \Theta$ and obtain $\Gamma, D_2\supset B \Rightarrow \Theta, \Delta$. Again by the inductive hypothesis, using $\Gamma \Rightarrow D_1, \Theta$, we get $\Gamma, D_1\supset(D_2\supset B) \Rightarrow \Theta, \Theta, \Delta$ and the conclusion follows by *Contr-R* and $L\&\supset$.

   (b) D is non-principal, with premises $\Gamma \Rightarrow D, E_1, \Theta'$ and $\Gamma \Rightarrow D, E_2, \Theta'$: use induction (twice) and then $R\&$.

3. If d ends with $R\vee$, there are two cases:

   (a) $D = D_1 \vee D_2$ is principal, with premise $\Gamma \Rightarrow D_1, D_2, \Delta$. By inductive hypothesis we get $\Gamma, D_1\supset B \Rightarrow D_2, \Theta, \Delta$. By another use of the inductive hypothesis we get $\Gamma, D_1\supset B, D_2\supset B \Rightarrow \Theta, \Delta, \Delta$, which we follow with a contraction on the right. (It is here that, in contrast to the single-succedent case, we are forced to use induction on the weight of D.) Then we use $L\vee\supset$.

   (b) D is not principal: use the inductive hypothesis and then $R\vee$.

4. If d ends with $R\supset$, there are two cases:

(a) $D = D_1 \supset D_2$ is principal, with premise $\Gamma, D_1 \Rightarrow D_2$. Weakening with $D_2 \supset B$ we obtain $\Gamma, D_2 \supset B, D_1 \Rightarrow D_2$; by $L\supset\supset$ the conclusion $\Gamma, (D_1 \supset D_2) \supset B \Rightarrow \Delta$ follows. Now weaken with Θ.

(b) D is not principal; then $\Gamma, E_1 \Rightarrow E_2$ for some $E_1 \supset E_2$ in Θ. So $\Gamma \Rightarrow E_1 \supset E_2$: now weaken with $D \supset B, \Delta$ and the rest of Θ.

5. If d ends with $L\supset\supset$, with $(F \supset G) \supset H$ principal and premises $\Gamma', G \supset H, F \Rightarrow G$ and $\Gamma', H \Rightarrow D, \Theta$ for $\Gamma = \Gamma', (F \supset G) \supset H$, we obtain

$$\dfrac{\dfrac{\Gamma', G \supset H, F \Rightarrow G}{\Gamma', G \supset H, F, D \supset B \Rightarrow G}\,W \qquad \dfrac{\Gamma', H \Rightarrow D, \Theta \qquad \dfrac{\Gamma', (F \supset G) \supset H, B \Rightarrow \Delta}{\Gamma', H, B \Rightarrow \Delta}\,Inv}{\Gamma', H, D \supset B \Rightarrow \Theta, \Delta}\,Ind}{\Gamma', (F \supset G) \supset H, D \supset B \Rightarrow \Theta, \Delta}\,L\supset\supset$$

□

Lemma 7.4 *The rule*

$$\frac{\Gamma \Rightarrow D \quad \Gamma, B \Rightarrow \Delta}{\Gamma, D \supset B \Rightarrow \Delta}$$

is admissible in **G4ip′**.

Proof: By Lemma 7.3, with Θ empty. □

Lemma 7.5 *The rule*

$$\frac{\Gamma, (C \supset D) \supset B \Rightarrow \Delta}{\Gamma, C, D \supset B, D \supset B \Rightarrow \Delta}$$

is admissible in **G4ip′**.

Proof: By induction on the height n of the derivation d of the premise. For $n = 0$, the premise is an axiom and so the conclusion is an axiom. If the last inference is by a right rule or by a left rule with $(C \supset D) \supset B$ non-principal, the induction is straightforward. If the last inference is by $L\supset\supset$ with $(C \supset D) \supset B$ principal, the premises are $\Gamma, C, D \supset B \Rightarrow D$ and $\Gamma, B \Rightarrow \Delta$; we now construct

$$\dfrac{\Gamma, C, D \supset B \Rightarrow D \qquad \dfrac{\Gamma, B \Rightarrow \Delta}{\Gamma, C, D \supset B, B \Rightarrow \Delta}\,W}{\Gamma, C, D \supset B, D \supset B \Rightarrow \Delta}\,\textit{Lemma 7.4}$$

□

Proposition 7.6 *The* Contraction-Left *rule*

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\,\textit{Contr-L}$$

is admissible in **G4ip′**.

Proof: By induction on the weight of the contracted formula A, and, for each weight, on the height n of the derivation d of the premise. For A atomic or $\bot$, if $n = 0$, then the premise and so also the conclusion are both axioms. If $n > 0$, then A is not principal in the premise; the conclusion follows by applying the inductive hypothesis to the premise(s) of the last step and then using the same rule as in the last inference of d.

For compound A, if $n = 0$, then $\bot \in \Gamma$ or Γ meets Δ in an atom and the conclusion follows. If $n > 0$ and A is not principal in the last inference of d, we apply the inductive hypothesis to the premises and then use the same rule as in the last inference of d. For principal A, if A is $C\&D$, $C \vee D$, $P\supset B$, $(C\&D)\supset B$ or $(C \vee D)\supset B$ we proceed in a uniform way by using an inversion lemma, contraction on lighter formulae, and then the same rule as in the last inference of d. For $A = (C\supset D)\supset B$ the derivation ends with

$$\frac{\Gamma, D\supset B, (C\supset D)\supset B, C \Rightarrow D \quad \Gamma, (C\supset D)\supset B, B \Rightarrow \Delta}{\Gamma, (C\supset D)\supset B, (C\supset D)\supset B \Rightarrow \Delta}$$

and by Lemma 7.5 the left premise gives $\Gamma, D\supset B, C, D\supset B, D\supset B, C \Rightarrow D$ and thus by applying three contractions, on the lighter formulae C and $D\supset B$, we obtain $\Gamma, D\supset B, C \Rightarrow D$. The right premise gives, by an inversion lemma and contraction on B, the sequent $\Gamma, B \Rightarrow E$. By $L\supset\supset$ the conclusion $\Gamma, A \Rightarrow \Delta$ follows. □

Proposition 7.7 *The rule*

$$\frac{\Gamma, A\supset B \Rightarrow A \quad \Gamma, B \Rightarrow \Delta}{\Gamma, A\supset B \Rightarrow \Delta}$$

is admissible in **G4ip′**.

Proof: By weakening the second premise with $A\supset B$, Lemma 7.4 and $Contr\text{-}L$. □

It follows that all the rules of **G3ip′** are admissible in **G4ip′**. The converse is easy (since *Cut* is admissible in **G3ip′**); so the two calculi are equivalent: but we do not use this in the proof of Theorem 7.8.

Theorem 7.8 *The* Cut *rule*

$$\frac{\Gamma \Rightarrow A, \Delta \quad \Gamma', A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\; Cut$$

is admissible in **G4ip′**.

Proof: By induction on the weight of A, with subsidiary inductions on the sum of the heights of the derivation of $\Gamma \Rightarrow A, \Delta$ and of the derivation of $\Gamma', A \Rightarrow \Delta'$. There are four cases:

1. At least one premise is an axiom;
2. Neither premise is an axiom and A is not principal in the first premise;
3. Neither premise is an axiom and A is principal in only the first premise;
4. Neither premise is an axiom and A is principal in both premises.

The first three cases are dealt with as in [4], as well as the fourth case for $A = B\&C$ and $A = B \vee C$. For example, in case 3, with $A = B\supset C$ and $\Gamma = \Gamma'', (F\supset G)\supset H$,

$$\dfrac{\dfrac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow A, \Delta}\,R\supset \quad \dfrac{A, \Gamma'', F, G\supset H \Rightarrow G \quad A, \Gamma'', H \Rightarrow \Delta'}{A, \Gamma'', (F\supset G)\supset H \Rightarrow \Delta'}\,L\supset\supset}{\Gamma, \Gamma'', (F\supset G)\supset H \Rightarrow \Delta, \Delta'}\,Cut$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow A}\,R\supset \quad A, \Gamma'', F, G\supset H \Rightarrow G}{\Gamma, \Gamma'', F, G\supset H \Rightarrow G}\,Cut \quad \dfrac{\dfrac{\Gamma, B \Rightarrow C}{\Gamma \Rightarrow A}\,R\supset \quad A, \Gamma'', H \Rightarrow \Delta'}{\Gamma, \Gamma'', H \Rightarrow \Delta'}\,Cut}{\Gamma, \Gamma'', (F\supset G)\supset H \Rightarrow \Delta'}\,L\supset\supset}{\Gamma, \Gamma'', (F\supset G)\supset H \Rightarrow \Delta, \Delta'}\,Weak\text{-}R}{}$$

.

In case 4, the subcase of implication splits into four further subcases:

1. $A = P\supset B$: The derivation ends as follows, where $\Gamma' = \Gamma'', P$:

$$\dfrac{\dfrac{\Gamma, P \Rightarrow B}{\Gamma \Rightarrow P\supset B, \Delta}\,R\supset \quad \dfrac{\Gamma'', P, B \Rightarrow \Delta'}{\Gamma', P\supset B \Rightarrow \Delta'}\,L0\supset}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

We transform this into

$$\dfrac{\dfrac{\dfrac{\Gamma, P \Rightarrow B \quad \Gamma'', P, B \Rightarrow \Delta'}{\Gamma, \Gamma'', P, P \Rightarrow \Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta'}\,Contr\text{-}L}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Weak\text{-}R$$

where the cut formula B is lighter than $P\supset B$.

2. $A = (C\&D)\supset B$: The derivation ends as follows:

$$\dfrac{\dfrac{\Gamma, C\&D \Rightarrow B}{\Gamma \Rightarrow (C\&D)\supset B, \Delta}\,R\supset \quad \dfrac{\Gamma', C\supset(D\supset B) \Rightarrow \Delta'}{\Gamma', (C\&D)\supset B \Rightarrow \Delta'}\,L\&\supset}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

We transform this into

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, C\&D \Rightarrow B}{\Gamma, C, D \Rightarrow B}\,Inv}{\Gamma, C \Rightarrow D\supset B}\,R\supset}{\Gamma \Rightarrow C\supset(D\supset B), \Delta}\,R\supset \quad \Gamma', C\supset(D\supset B) \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

where the cut formula $C\supset(D\supset B)$ is lighter than $(C\&D)\supset B$.

3. $A = (C \vee D)\supset B$: Similar to the previous cases, using the inversion lemma for $L\vee$, $R\supset$, *Cut* (twice) on lighter formulae and left contractions on Γ.

4. $A = (C\supset D)\supset B$: The derivation ends with

$$\dfrac{\dfrac{\Gamma, C\supset D \Rightarrow B}{\Gamma \Rightarrow (C\supset D)\supset B, \Delta}\,R\supset \qquad \dfrac{\Gamma', D\supset B, C \Rightarrow D \quad \Gamma', B \Rightarrow \Delta'}{\Gamma', (C\supset D)\supset B \Rightarrow \Delta'}\,L\supset\supset}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Cut$$

and is transformed into the following derivation with four cuts on the lighter formulae $C\supset D$, $D\supset B$ and B:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{D, C \Rightarrow D}\,7.1}{D \Rightarrow C\supset D} \quad \Gamma, C\supset D \Rightarrow B}{\Gamma, D \Rightarrow B}\,Cut}{\Gamma \Rightarrow D\supset B}\,R\supset \quad \Gamma', D\supset B, C \Rightarrow D}{\Gamma, \Gamma', C \Rightarrow D}\,Cut}{\Gamma, \Gamma' \Rightarrow C\supset D}\,R\supset \quad \Gamma, C\supset D \Rightarrow B}{\Gamma, \Gamma, \Gamma' \Rightarrow B}\,Cut \quad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow \Delta'}\,Cut}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\,Contr, W$$

.

□

Note again that we could have used more cuts in place of the appeal to *Inv*.

Following [4], we have chosen a particular form of the rule $L\supset\supset$ without the succedent of the conclusion appearing in the first premise. There is a possible variant, summed up in

Proposition 7.9 *The rule* $L\supset\supset''$

$$\dfrac{D\supset B, \Gamma \Rightarrow C\supset D, \Delta \quad B, \Gamma \Rightarrow \Delta}{(C\supset D)\supset B, \Gamma \Rightarrow \Delta}\,L\supset\supset''$$

is admissible in **G4ip$'$**.

Proof: In **G4ip$'$+Cut** we have

$$\dfrac{\dfrac{(C\supset D)\supset B \Rightarrow D\supset B \qquad \dfrac{D\supset B, \Gamma \Rightarrow C\supset D, \Delta \qquad \dfrac{(C\supset D)\supset B, C\supset D \Rightarrow B \quad B, \Gamma \Rightarrow \Delta}{(C\supset D)\supset B, C\supset D, \Gamma \Rightarrow \Delta}\,Cut}{D\supset B, (C\supset D)\supset B, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\,Cut}{(C\supset D)\supset B, (C\supset D)\supset B, \Gamma, \Gamma \Rightarrow \Delta, \Delta}\,Cut}{(C\supset D)\supset B, \Gamma \Rightarrow \Delta}\,Contr$$

from which the cuts and contractions can now be eliminated. The first and third premises are easily derived using *Cut*, $R\supset$ and Lemma 7.1. □

Corollary 7.10 **G4ip$'$** *is equivalent to* **G4ip$''$**, *the calculus with the rules of* **G4ip$'$** *except for* $L\supset\supset''$ *in place of* $L\supset\supset$.

Proof: By the proposition, every rule of **G4ip$''$** is admissible in **G4ip$'$**. Conversely, it is easy to see that $L\supset\supset$ is admissible in **G4ip$''$**. □

Example: The rule

$$\dfrac{C, D\supset B, \Gamma \Rightarrow D, \Delta \quad B, \Gamma \Rightarrow \Delta}{(C\supset D)\supset B, \Gamma \Rightarrow \Delta}\,L\supset\supset'''$$

is not admissible in **G4ip$'$**, by consideration of the non-derivable sequent $(p\supset q)\supset r, r\supset p \Rightarrow p$.

8 Extension with quantifiers

We consider the extension **G4i** of **G4ip** obtained by adding the quantifiers. Quantified formulae are weighted by $w(\forall x A) := 1 + w(A)$ and $w(\exists x A) := 2 + w(A)$. Besides the usual rules

$$\frac{\Gamma, \forall x A x, At \Rightarrow E}{\Gamma, \forall x A x \Rightarrow E} L\forall \quad \frac{\Gamma \Rightarrow Ay}{\Gamma \Rightarrow \forall x A x} R\forall$$

$$\frac{\Gamma, Ay \Rightarrow E}{\Gamma, \exists x A x \Rightarrow E} L\exists \quad \frac{\Gamma \Rightarrow At}{\Gamma \Rightarrow \exists x A x} R\exists$$

for $\forall$ and $\exists$ we have the rules for the refinement of the $L\supset$ rule in the cases where the antecedent of the principal formula is quantified:

$$\frac{\Gamma, \forall x A x \supset B \Rightarrow \forall x A x \quad \Gamma, B \Rightarrow E}{\Gamma, \forall x A x \supset B \Rightarrow E} L\forall\supset \quad \frac{\Gamma, \forall x (A x \supset B) \Rightarrow E}{\Gamma, \exists x A x \supset B \Rightarrow E} L\exists\supset$$

The usual restrictions (cf. [4]) on variables apply for $R\forall$ and $L\exists$.

In order to extend admissibility of contraction of **G4ip** to **G4i** we need the inversion lemmas for each left rule that does not duplicate its principal formula into the premise(s).

Lemma 8.1 *The following rules are strongly admissible in* **G4i***:*

1. $\dfrac{\Gamma, (\forall x A x) \supset B \Rightarrow E}{\Gamma, B \Rightarrow E}$;
2. $\dfrac{\Gamma, \exists x A x \supset B \Rightarrow E}{\Gamma, \forall x (A x \supset B) \Rightarrow E}$;
3. $\dfrac{\Gamma, \exists x A x \Rightarrow E}{\Gamma, At \Rightarrow E}$

Proof: Routine. □

Lemma 8.2 *Judgments of the following form*

1. $\Gamma, A \Rightarrow A$ *(generalized axiom)*
2. $\Gamma, A, A \supset B \Rightarrow B$ *(modus ponens)*

are derivable in **G4i***.*

Proof: 1. Following the proof of Lemma 3.2, we consider the extra cases with A of the form $\forall x G x$, $\exists x G x$, $\forall x G x \supset C$ and $\exists x G x \supset C$: By induction we have $\Gamma, Gx \Rightarrow Gx$ (choosing x not free in Γ), from which the conclusions $\Gamma, \forall x G x \Rightarrow \forall x G x$ and $\Gamma, \exists x G x \Rightarrow \exists x G x$ follow by obvious logical steps. For $A = \forall x G x \supset C$ we have the derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \forall x G x \supset C, \forall x G x \Rightarrow \forall x G x} Ind \quad \dfrac{}{\Gamma, C, \forall x G x \Rightarrow C} Ind}{\Gamma, \forall x G x \supset C, \forall x G x \Rightarrow C} L\forall\supset}{\Gamma, \forall x G x \supset C \Rightarrow \forall x G x \supset C} R\supset$$

and for $A = \exists x Gx \supset C$ we have the derivation (choosing x not free in Γ, C)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \forall x(Gx\supset C), Gx\supset C \Rightarrow Gx\supset C}\;Ind}{\Gamma, \forall x(Gx\supset C), Gx\supset C, Gx \Rightarrow C}\;3.1, 8}{\Gamma, \forall x(Gx\supset C), Gx \Rightarrow C}\;L\forall}{\Gamma, \forall x(Gx\supset C), \exists x Gx \Rightarrow C}\;L\exists}{\Gamma, \exists x Gx\supset C, \exists x Gx \Rightarrow C}\;L\exists\supset}{\Gamma, \exists x Gx\supset C \Rightarrow \exists x Gx\supset C}\;R\supset$$

2. Follows from 1 as in the proof of lemma 3.2. □

Then we have:

Proposition 8.3 *The* Contraction *rule is admissible in* **G4i***.*

Proof: We only have to add in the proof of admissibility of *Contraction* for **G4ip** the following cases:

$A = \forall x Cx \supset B$: The derivation ends with

$$\dfrac{\Gamma, \forall x Cx\supset B, \forall x Cx\supset B \Rightarrow \forall x Cx \quad \Gamma, \forall x Cx\supset B, B \Rightarrow E}{\Gamma, \forall x Cx\supset B, \forall x Cx\supset B \Rightarrow E}\;L\forall\supset$$

By inductive hypothesis the left premise gives $\Gamma, \forall x Cx\supset B \Rightarrow \forall x Cx$; by the (partial) inversion lemma 8.1 for $L\forall\supset$ and the inductive hypothesis the right premise gives $\Gamma, B \Rightarrow E$ and the conclusion follows by $L\forall\supset$.

$A = \exists x Cx \supset B$: The derivation ends with

$$\dfrac{\Gamma, \exists x Cx\supset B, \forall x(Cx\supset B) \Rightarrow E}{\Gamma, \exists x Cx\supset B, \exists x Cx\supset B \Rightarrow E}\;L\exists\supset$$

By the inversion lemma for $L\exists\supset$ the premise gives $\Gamma, \forall x(Cx\supset B), \forall x(Cx\supset B) \Rightarrow E$, hence the conclusion follows by applying contraction on the lighter formula $\forall x(Cx\supset B)$ and then the rule $L\exists\supset$.

The cases with $A = \forall x Cx$ and $A = \exists x Cx$ are dealt with as in [26] or [4]. □

Theorem 8.4 *The* Cut *rule is admissible in* **G4i***.*

Proof: In the proof of Theorem 6.1 we consider the extra cases in which the cut formula A is principal in both premises and it is $\forall x Cx\supset D$ or $\exists x Cx\supset D$ or $\forall x Cx$ or $\exists x Cx$.

$A = \forall x Cx \supset D$: The derivation

$$\dfrac{\dfrac{\Gamma, \forall x Cx \Rightarrow D}{\Gamma \Rightarrow \forall x Cx\supset D}\;R\supset \qquad \dfrac{\Gamma', \forall x Cx\supset D \Rightarrow \forall x Cx \quad \Gamma', D \Rightarrow E}{\Gamma', \forall x Cx\supset D \Rightarrow E}\;L\forall\supset}{\Gamma, \Gamma' \Rightarrow E}\;Cut$$

is transformed into

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \forall x Cx\supset D \quad \Gamma', \forall x Cx\supset D \Rightarrow \forall x Cx}{\Gamma, \Gamma' \Rightarrow \forall x Cx}\;Cut \qquad \Gamma, \forall x Cx \Rightarrow D}{\Gamma, \Gamma, \Gamma' \Rightarrow D}\;Cut \qquad \Gamma', D \Rightarrow E}{\Gamma, \Gamma, \Gamma', \Gamma' \Rightarrow E}\;Cut}{\Gamma, \Gamma' \Rightarrow E}\;Contr$$

using one cut on A with right premise of smaller height and two cuts on lighter formulae.

$A = \exists x Cx \supset D$: The derivation

$$\dfrac{\dfrac{\Gamma, \exists x Cx \Rightarrow D}{\Gamma \Rightarrow \exists x Cx \supset D} R\supset \quad \dfrac{\Gamma', \forall x(Cx \supset D) \Rightarrow E}{\Gamma', \exists x Cx \supset D \Rightarrow E} L\exists\supset}{\Gamma, \Gamma' \Rightarrow E} Cut$$

is transformed, using the inversion lemma for $L\exists$, into

$$\dfrac{\dfrac{\dfrac{\Gamma, Cy \Rightarrow D}{\Gamma \Rightarrow Cy \supset D} R\supset}{\Gamma \Rightarrow \forall x(Cx \supset D)} R\forall \quad \Gamma', \forall x(Cx \supset D) \Rightarrow E}{\Gamma, \Gamma' \Rightarrow E} Cut$$

so the cut is replaced by a cut on a lighter formula.

The cases with $A = \forall x Cx$ and $A = \exists x Cx$ are dealt with as in [26] or [4]. □

9 Related work

As mentioned in the introduction, there are many papers showing the admissibility of structural rules for **G4ip**, beginning with [28] for a related system. Almost all of these use an induction on sequent weight, which does not extend easily to calculi with a rule with a premiss of weight no less than the conclusion, such as the extensions of **G4ip** with apartness or quantifiers.

In particular, Hudelmaier [10], [11], [12] argues for the admissibility of the structural rules in **G4ip** by a combination of inductions on derivation height and on sequent weight and a reduction to the admissibility result for **G3ip**. [11] and [12] also include fast cut-elimination procedures for **G3ip**. [6, 15, 26, 29] argue similarly. In [12], the proof of admissibility in **G4ip** of the standard **G3i** rule $L\supset$ for left-introduction of implication with $A \supset B$ principal only works (but is only needed [23]) for atomic A. This error was pointed out [14] by Gordeev. There is another proof of completeness of **G4ip** in [11]; this uses a different approach, essentially a proof that every derivation in **G3ip** can be "purified", together with an induction on sequent weight. (A **G3ip**-derivation is "pure" when every instance of $L\supset$ has as left premiss either an axiom or a derivation ending in an R-rule.) A variant of this proof may be interpreted as an algorithm for cut elimination in **G4ip**, with a Kalmar-elementary upper bound on the growth of proof depth; but, as remarked in the last sentence of [11], this bound is unrealistic in view of the much sharper upper bound on depth of cut-free proofs given by the special properties of **G4ip**.

The referee has suggested that Hudelmaier's cut-elimination procedure [11] is to be preferred, being "faster" and therefore better. We are not aware of any implementation of any of his cut-elimination procedures, other than the procedure of throwing away the proof and searching for a cut-free proof of the same end-sequent. (As already mentioned, there are several implementations of the cut-free calculi as proof search procedures.) Moreover, we do not agree that speed of a cut-elimination procedure is the sole determinant of the usefulness of the procedure; simplicity and easy extensibility are also worth having.

[10] handles the first-order case by an induction on a combination of sequent weight and the number of "bad" instances of $L\supset$, i.e. instances where the principal formula really must be

retained in the left premise. Such a technique would, we believe, also work for extensions with (e.g.) apartness or order; however, the approach even in the zero-order case is considerably more complex than our own.

Miglioli *et al* give a semantic proof [17] of related results. Rules slightly different from those of **G4ip** are used, with extra complexities arising from a primitive negation operator and with, for example, the invertible rule $L0\supset$ replaced by a non-invertible rule. Such proofs need substantial technical machinery about Kripke models, in contrast to the routine constructive nature of the present direct proof. With difficulty, the proof extends to the first-order case in [17].

Weich [30] has recently given a constructive proof that every formula has either a natural deduction proof or a counter-model; from a machine-verified version of this proof, a SCHEME implementation of a proof search algorithm has been extracted. This algorithm constructs a proof in **G4ip** (or, rather, Hudelmaier's improvement thereon in [13]).

Finally, it is appropriate to mention here that the joint paper [21] of the first author with Luis Pinto, on the relationship between the multisuccedent calculus **G4ip′** and Kripke models, contains a minor error (first pointed out by Uwe Egly): the paper is mainly but not entirely based on the convention that f (alias $\bot$) is an atomic formula, and thus it is not clear how the rules for the refutation calculus **CRIP** deal with formulae of the form $f\supset B$ and f in the antecedent. With the convention that f is not an atomic formula, as in the present paper, two solutions are given in [19]. With the opposite convention, it should be stated in [21] that the proviso on the rule (11) should be extended by the proviso that $f \notin \Gamma$. One may dispense with the *axiom* rule—it is the special case of (11) when $n + m = 0$.

10 Conclusion

We have given a direct proof of admissibility of the usual structural rules for the sequent calculus **G4ip** and shown how to extend the proof when the calculus is extended with first-order syntax and with multiple succedents.

The problem originally arose from a question of conservativity of apartness over equality (defined as the negation of apartness) and of theories based on excess over theories based on partial order (defined as the negation of excess), as formulated in [27]. This problem led to the need to extend *Cut* admissibility from logical calculi to calculi with non-logical rules of inference in which the weights of the premises may be greater than the weight of the conclusion [18].

Here we have shown admissibility of the structural rules for the basic contraction-free systems of intuitionistic logic based on **G4ip**. In a sequel we will deal with extensions: non-logical rules for the theories of apartness and order, Dummett logic, lax logic and Kuroda logic.

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