

Proof-theoretic harmony and the meaning of the classical constants

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Gentzen's remarks

- ▶ *The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. (Gentzen, 1934, p. 80)*
- ▶ *By making these ideas more precise it should be possible to display the E-inferences as unique functions of their corresponding I-inferences, on the basis of certain requirements. (Ibid.)*

My plan today

- ▶ How to make Gentzen's ideas more precise?
- ▶ My **plan** today:
 - ▶ I'll say something about the idea that basic inference rules **define meanings**;
 - ▶ I'll say something about **harmony**;
 - ▶ I'll argue that some of the standard ways for applying Gentzen's ideas to **classical logic** are problematic, and I'll suggest how one may fix them up.

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Two desiderata

- ▶ One should be able to **justify** an account of proof-theoretic harmony.
- ▶ I-rules shouldn't have **semantic priority** over the corresponding E-rules, and *vice versa*: I- and E-rules

are alternative in that either is sufficient to determine the meaning of a sentence uniquely. (Dummett, 1993, p. 142)

- ▶ Harmonious I- and E-rules should uniquely determine, respectively, the corresponding E- and I-rules:

any introduction rule, taken on its own, succeeds in conferring on its featured connective a precise logical sense. That sense in turn dictates what the corresponding elimination rule must be. Mutatis mutandis, any elimination rule, taken on its own, succeeds in conferring on its featured connective a precise logical sense. That sense in turn dictates what the corresponding introduction rule must be. (Tennant, 2005, 628)

Necessary and sufficient conditions

- ▶ My starting point will be the thought that I-rules define the meanings of the constants they introduce in virtue of specifying **necessary and sufficient conditions** for assertion.
- ▶ Thus **Read** (but see also Schroeder-Heister's **definitional closure**):

*what is implicit in the totality of cases of the introduction-rule for a connective is that they **exhaust the grounds for assertion** of that specific conclusion.*

*Introduction and elimination [...] rules **already contain both necessary and sufficient conditions for assertion** [...] and it is when these do not agree that problems like those of 'tonk' arise (Read, 2008, p. 6)*

The Fundamental Assmption

- ▶ It seems plausible to require that I-rules collectively specify **necessary conditions** in the following sense:

The introduction rules for a constant \$ represent the direct or canonical means of establishing the truth of a sentence with principal operator \$. A statement may frequently be established by indirect means, but to label certain means 'canonical' is to claim that, whenever we are justified in asserting the statement, we could have arrived at our entitlement to do so by those restricted means. [...] If a statement whose principal operator is one of the logical constants in question can be established at all, it can be established by an argument ending with one of the stipulated introduction rules. (Dummett, 1991, p. 252)

- ▶ This is Dummett's and Prawitz's **Fundamental Assumption (FA)**.
- ▶ Can we justify, and perhaps define, **harmony** on this assumption?

Prawitz's Inversion Principle

- ▶ Well, if complex statement can **only** be introduced by means of its I-rules, then it also seems plausible to require that

by an elimination rule one [...] restores what had already been established by the major premiss of the application of an introduction rule. (Prawitz, 1965, p. 33)

- ▶ This yields Prawitz's **Inversion Principle**:

let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition [...] for deriving the major premiss of α , when combined with deductions of the minor premisses of α (if any), already "contain" a deduction of B ; the deduction of B is thus obtainable directly from the given deductions without the addition of α . (Ibid.)

- ▶ One may then say that the $\$$ -I and $\$$ -E are in **harmony** if **maximum formulae** of the form $\$(A_1, \dots, A_n)$ can always be removed.

(At most) one half of the story

- ▶ Prawitz's principle guarantees that

nothing is gained by inferring a formula through introduction for use as a major premiss in an elimination. (Prawitz, 1965, pp. 34-5)

- ▶ However, reducibility alone does not guarantee that E-rules be **uniquely determined** by the I-rules, as Prawitz well knows:

Although it is clear that the elimination rules must satisfy certain requirements which were expressed in terms of an inversion principle [...], it is not clear in what sense (if at all) these properties determine the elimination rules uniquely. (Prawitz, 1971)

- ▶ Let \circ be a connective governed by \wedge -I and half of \wedge -E.
- ▶ Local peaks, i.e. **maximum formulae**, of the form $A \circ B$ can be removed, or levelled, and yet the E-rule is clearly too weak.

The Generalized Inversion Principle

- ▶ Perhaps a better way to make Gentzen's ideas **more precise** is to require that

whatever follows from the canonical grounds for α , i.e. from the necessary and sufficient conditions for asserting α , also follows from α (see Negri and von Plato, 2001).

- ▶ Call this the **Generalized Inversion Principle**.
- ▶ Assume that γ follows from the canonical grounds for asserting α . Now assume α . If α can **only** be inferred from its canonical grounds, γ must follow from α itself, given our assumption that it follows from the canonical grounds for α .
- ▶ The principle is often associated with a **schema** for deriving harmonious GE rules; see e.g. Read (2000).

The GE schema

- ▶ Take an arbitrary introduction rule:

$$\frac{\beta_1^1, \dots, \beta_{n_1}^1}{\alpha} \dots \frac{\beta_1^k, \dots, \beta_{n_k}^k}{\alpha} \alpha\text{-I}$$

- ▶ Then, the corresponding E-rule is to be found as follows:

$$\frac{\underbrace{[\beta_1^1, \dots, \beta_{n_1}^1]^{(j)}}_{\vdots} \quad \dots \quad \underbrace{[\beta_1^k, \dots, \beta_{n_k}^k]^{(j)}}_{\vdots}}{\frac{\alpha}{\gamma} \quad \underbrace{\hspace{10em}}_{\alpha\text{-E, } j}} \gamma$$

- ▶ One may then say that I- and E-rules are in **harmony** if the E-rules are derived by means of the above procedure.

The GE schema and the Generalized Inversion Principle

- ▶ Here is a [first potential concern](#), though.
- ▶ [Moriconi and Tesconi](#) argue that the GE schema isn't justified by the Generalized Inversion Principle.
- ▶ They write:

In our opinion, [...] general elimination rules state much more than simply the principle to which they should correspond. [...] Negri and von Plato's general elimination rules entail their correctness because the hypotheses [they] depend on are made 'internal' to the rules. This is done by turning those hypotheses into premisses of the rules themselves. (Moriconi and Tesconi, 2008, p. 112)

Symmetry

- ▶ One of our desiderata was that E-rules be derivable from the corresponding I-rules **and vice versa**.
- ▶ I take this to mean that one should be able to derive harmonious E-rules from **arbitrary canonical grounds**, and that, likewise, one should be able to derive harmonious I-rules from **arbitrary canonical consequences**.
- ▶ However, GE harmony doesn't give us that: one can only derive I-rules from GE-rules by, so to speak, extracting canonical grounds from the rules.
- ▶ It seems to me that canonical grounds have **conceptual priority**, and the notion of a **canonical consequence** plays no distinctive role.

Weir on \forall -E and \exists -E

- ▶ Consider the standard E-rules for \forall and \exists .
- ▶ Unlike the standard E-rules for the remaining constants, they typically don't allow us to reintroduce the complex statements they allow us to eliminate.
- ▶ Thus Alan Weir:
 - ▶ "C need not be identical with A_i , the premise of the original application of \forall -I" (Weir, 1986, p. 464);
 - ▶ "one cannot derive, by \exists -I alone, $\exists xF(x)$ from an arbitrary conclusion, C , of $\exists E$ " (Weir, 1986, p. 471);
 - ▶ E-rules for \forall and \exists do not extract all that is contained in their corresponding I-rules: there is some leakage (see Weir, 1986, p. 467).

Sequent calculus and natural deduction

- ▶ Now, symmetry can (perhaps) be achieved in a [sequent calculus](#) setting.
- ▶ Thus Wagner [Campos Sanz](#) and Thomas [Piecha](#) in a recent paper:

Left introduction rules are inverses of the right introduction rules in the precise sense of being admissible given the right introduction rules as primitive. [...] We can also show the converse direction, viz. that the right introduction rules are admissible given the left introduction rules as primitive. This is due to the [inherent symmetry of the sequent calculus](#). This symmetry is not present in natural deduction, which makes the formulation of an inversion principle based on elimination rules rather than introduction rules [quite difficult](#). (Campos Sanz and Piecha, 2009, p. 551)

- ▶ But what about [natural deduction](#) rules?

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Classical inferentialism

- ▶ Things get even **worse** if one wishes to use a natural deduction framework *and* be a **classical inferentialist**.
- ▶ The problem is not so much with **standard rules for negation**, which are not harmonious (on any minimally decent account of harmony), and hence defective, on the foregoing assumptions.
- ▶ For there *are* natural deduction harmonious formalizations of classical logic:
 - ▶ **Borh**ic (1985) and Read (2000) give a harmonious natural deduction, **multiple conclusions** formalization of classical logic;
 - ▶ **Smiley** (1996) and Rumfitt (2000) give a harmonious natural deduction **bilateralist** formalization of classical logic.
- ▶ Rather, the problem is that, in both systems, the proof of LEM violates our assumption that I-rules specify **necessary conditions for assertion**.

Bilateralism and multiple conclusions systems

- ▶ Drawing on Smiley (1991, p. 5), Ian Rumfitt (2000) presents a **bilateral** formalization of classical logic, where '+' and '-' are nonembeddable force signs, and '+A' and '-A' are **signed formulae** for any $A \in WFF$, indicating "A? Yes" and "A? No" respectively.
- ▶ The system has the standard structural rules, Reflexivity, Dilution and Cut, together with the following two 'coordination principles': the following form of *reductio*,

(RED*) From $\alpha \vdash \perp$, infer $\vdash \alpha^*$,

and the following form of the Law of Non-Contradiction,

(LNC*) From α, α^* , infer \perp ,

where lower case greek letters range over signed formulae, and α^* is the result of reversing α 's sign.

- ▶ The **operational rules** are harmonious and include both rules for asserting and rules for denying complex statements.
- ▶ **Natural deduction multiple conclusions** formalizations of classical logic need no introduction.

A bad result

- ▶ I take this to be a **bad result** for the classical inferentialist.
- ▶ If I-rules, or canonical grounds, are not assumed to contain, in some sense, **necessary conditions** for assertion, it is difficult to see:
 - ▶ how rules can **define** meanings;
 - ▶ how harmony can be **justified** in the first place.

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Two problems, a two-steps solution

- ▶ Here is something the **classical bilateral inferentialist** might do:
 - (i) Following **Lorenzen**, solve the **symmetry problem** by defining harmony as the requirement that E-rules give us back **precisely** the canonical grounds specified by the corresponding I-rules;
 - (ii) Find suitable fully invertible rules for \vee and \exists : rules that are harmonious in the proposed symmetric sense, and that respect Dummett's and Prawitz's **Fundamental Assumption**.

Lorenzen's *Inversionprinzip*

- ▶ Let $\mathcal{CG}[A]$ be A 's **canonical grounds**, and suppose that they specify the necessary and sufficient conditions for asserting A .
- ▶ Assume that, for some B , the rule $\mathcal{CG}[A] \Rightarrow B$ is admissible. Now assume A . Since A can **only** be introduced, *ex hypothesi*, by means of its canonical grounds, $\mathcal{CG}[A]$ follows. But then, B also follows. Hence, the rule $A \Rightarrow B$ is admissible if the rule $\mathcal{CG}[A] \Rightarrow B$ is also admissible.

$$\frac{\mathcal{CG}[A] \Rightarrow B}{A \Rightarrow B} \text{ (Inversionprinzip)}$$

- ▶ Now set $B := \mathcal{CG}[A]$. Then, since obviously $\mathcal{CG}[A] \Rightarrow \mathcal{CG}[A]$, $\mathcal{CG}[A] \Rightarrow A$ only if $A \Rightarrow \mathcal{CG}[A]$ (see [Moriconi and Tesconi, 2008](#), p. 109).

Two inversion principles

- ▶ We end up with **two** inversion principles. The first is briefly described by **Negri and von Plato**:

*a more **limited inversion principle**, one requiring that **elimination rules conclude the immediate grounds for deriving a proposition** instead of arbitrary consequences of these grounds. (Negri and von Plato, 2001, p. 9)*

- ▶ If $\mathcal{CG}[A] \Rightarrow A$, then $A \Rightarrow \mathcal{CG}[A]$.
- ▶ The **second limited inversion principle** allows us to deriving I-rules from E-rules, and requires that **complex sentences be immediately inferable from their immediate consequences**.
- ▶ If $A \Rightarrow \mathcal{CC}[A]$, then $\mathcal{CC}[A] \Rightarrow A$.
- ▶ **Both** I- and E-rules contain necessary and sufficient conditions for assertion, “and it is when these don't agree that problems like **tonk** arise”.

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Moriconi and Tesconi's objection

- ▶ Lorenzen's procedure depends upon the possibility of finding some B such that the rule $\mathcal{CG}[A] \Rightarrow B$ is always admissible.
- ▶ But this isn't always possible: as [Moriconi and Tesconi](#) point out, the rules for \forall and \exists don't respect this requirement.
- ▶ Moriconi and Tesconi conclude that
the reasoning described is unsatisfactory because it works in theory but not in practice. (Moriconi and Tesconi, 2008, p. 109)

An alternative diagnosis

- ▶ But what if, *pace* Moriconi and Tesconi, something like Lorenzen's reasoning can work?
- ▶ That is, what if the problem rather lies with the **standard rules** for \vee and \exists ?
- ▶ We've already seen that
 - (i) these rules are validated by a conception of harmony that doesn't satisfy our symmetry requirement,
 - (ii) the rules for \vee don't satisfy FA, if our aim is to justify **classical logic**.

Classical disjunction

- Consider the following rules:

$$\begin{array}{c}
 \vdots \\
 \perp \\
 +\forall\text{-I}^*, i \frac{\perp}{+\forall\text{-I}^*, i} \\
 +\forall\text{-E}^* \frac{+\forall\text{-I}^*, i \quad +A \vee B \quad -A \quad -B}{\perp}
 \end{array}$$

- These rules are **fully invertible**: an application of $\forall\text{-I}^*$ immediately followed by an application $\forall\text{-E}^*$ returns us to the canonical grounds $\forall\text{-I}^*$, and *vice versa*.
- Local peaks** of the form $A \vee B$ can always be levelled:

$$\begin{array}{c}
 \Delta, [-A, -B]^i \\
 \Sigma \\
 +\forall\text{-I}^*, i \frac{\perp}{+\forall\text{-I}^*, i} \\
 +\forall\text{-E}^* \frac{+\forall\text{-I}^*, i \quad +A \vee B \quad \Gamma_0 \quad \Gamma_1}{\perp}
 \end{array}
 \quad \rightsquigarrow \quad
 \begin{array}{c}
 \Gamma_0 \quad \Gamma_1 \\
 \Pi_0 \quad \Pi_1 \\
 \Delta \quad -A \quad -B \\
 \underbrace{\hspace{10em}} \\
 \Sigma \\
 \perp
 \end{array}$$

- LEM has a **canonical proof**: Assume $+A$ and $+\neg A$. By $-\neg\text{-I}$, infer $- \perp$. Contradiction. By LNC*, infer \perp . By $\forall\text{-I}^*$, discharge $+A$ and $+\neg A$, and conclude $+A \vee \neg A$. ■

Classical existence

- As for the **existential quantifier**, classical logicians may adopt the following rules:

$$\begin{array}{c}
 \vdots \\
 \perp \\
 +\exists\text{-I, } i \frac{\quad}{+\exists x Fx} \\
 \Gamma_0 \qquad \qquad \Gamma_1 \\
 \vdots \qquad \qquad \qquad \vdots \\
 +\exists\text{-E} \frac{+\exists x Fx \qquad -F(a/x)}{\perp}
 \end{array}$$

where a may not occur free in $\exists x Fx$ and Γ_1 .

- The rules for **denying existential statements** are just like the ones for asserting them, except that the signs are inverted.
- These rules are **fully invertible** and reducible, as shown by the following reduction:

$$\begin{array}{c}
 \Gamma_0, [-F(t/x)]^i \\
 \Pi_0 \\
 +\exists\text{-I, } i \frac{\quad}{+\exists x Fx} \\
 +\exists\text{-E} \frac{\quad}{\perp} \\
 \Gamma_1 \\
 \Pi_1 \\
 -F(a/x) \\
 \Gamma_0, -F(t/x) \\
 \Pi_0 \\
 \perp
 \end{array}
 \rightsquigarrow
 \begin{array}{c}
 \Gamma_1 \\
 \Pi_1 \\
 -F(a/x) \\
 \Gamma_0, -F(t/x) \\
 \Pi_0 \\
 \perp
 \end{array}$$

What we (can) get

- ▶ It can be shown that the standard bilateral rules for \forall and \exists are derivable from my proposed ones, and vice versa.
- ▶ What we get, then, is a sound and complete formalization of **classical logic** that satisfies the following properties:
 - (i) the rules are harmonious, in the sense that I- and E-rules are **interderivable** (via the restricted inversion principle and its **converse**);
 - (ii) the system **normalizes** (one can prove, as a corollary, that the system is also **separable** and has the **sub-formula property**);
 - (iii) the **Fundamental Assumption** can be assumed even in a classical setting, which means that the inferentialist's main justification for harmony, that both I- and E-rules contain sufficient but also **necessary** conditions for asserting complex statements, is still valid.

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Weir 1986

- ▶ Alan Weir (1986) suggests the following **bipartite Inversion Principle** on behalf of the classical inferentialist.
- ▶ As I show, however, Weir's principle, albeit related to the one I wish to eventually recommend, is nevertheless **defective**, as it offers no protection from Prior's tonk and its ilk.
- ▶ Weir introduces the **first half** of the principle thus:

When the [necessary and] sufficient condition for application of an I-rule obtains, application of that rule followed immediately by application of elimination rules for the relevant constant returns us to the [necessary and] sufficient condition for application of the I-rule. (Weir, 1986, p. 466)

Weir's Inversion Principle: first half (I-E)

$$\blacktriangleright \quad \frac{\Pi}{*(C_1, \dots, C_n)} := \frac{\begin{array}{c} [A_1^i] \quad [A_1^i] \\ \vdots \quad \vdots \\ P_1^i \quad P_{r(i)}^i \end{array}}{*(C_1, \dots, C_n)} \star\text{-I} \quad \rightsquigarrow$$

$$\blacktriangleright \quad \frac{\Pi}{*(C_1, \dots, C_n)} \frac{[A_1^i] \dots [A_{r(i)}^i]}{P_1^i} \star\text{-E} \quad \dots \quad \frac{\Pi}{*(C_1, \dots, C_n)} \frac{[A_1^i] \dots [A_{r(i)}^i]}{P_{r(i)}^i} \star\text{-E}$$

Weir 1986

- ▶ Weir supplements the first half of his principle by a second half, to the effect that E-rules should be strong enough to allow us to immediately reintroduce the complex statement they allow us to eliminate.
- ▶ In [Weir's](#) own words:

When the sufficient condition for application of an E-rule obtains, application of that rule followed immediately by application of introduction rules for the relevant constant returns us to the sufficient condition for application of the E-rule. (Weir, 1986, p. 467)

Weir's Inversion Principle: second half (E-I)

$$\triangleright \frac{\prod E_i \quad \star(C_1, \dots, C_n) \quad \begin{array}{c} [A_1^i] \\ \vdots \\ M_1^i \end{array} \quad \begin{array}{c} [A_{r(i)}^i] \\ \vdots \\ M_{r(i)}^i \end{array}}{E_i} \star\text{-E} \quad \rightsquigarrow$$

$$\triangleright \frac{\frac{\prod [M_1^1]^j [M_r^1(i)]^j}{E_1} \star\text{-E} \quad \dots \quad \frac{\prod [M_1^s]^j [M_r^s(i)]^j}{E_s} \star\text{-E}}{\star(C_1, \dots, C_n)} \star\text{-I, } j$$

What went wrong

- ▶ A little reflection suffices to show that Weir's [Inversion Principle](#) validates the following [rules](#):

$$\text{▶ } \frac{A}{A \oplus B} \oplus\text{-I}_1 \quad \frac{B}{A \circ B} \oplus\text{-I}_2$$

$$\text{▶ } \frac{A \oplus B}{A} \oplus\text{-E}_1 \quad \frac{A \oplus B}{B} \oplus\text{-I}_2$$

- ▶ These rules are [pairwise harmonious](#), on any decent account of harmony.
- ▶ However, they collectively yield [disaster](#).