Gentzen's logical calculi

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Gentzen Centenary
St Andrews, 24.XI.09

Motto: „Spannend wie ein Kriminalroman!“

The aim

I have set as my specific task a proof of the consistency of logical inference in arithmetic... The task becomes a purely mathematical problem through the formalization of logical inference. The proof of consistency has been so far carried out only for special cases, for example, the arithmetic of the integers without the rule of complete induction. I would like to proceed further at this point and to clear at least arithmetic with complete induction. I am working on this since almost a year and hope to finish soon, and would then present this work as my dissertation (with Prof. Bernays).

Gentzen, letter to Hellmuth Kneser, 13 Dec. 1932

Some dates

- Work on **Hertz systems** in the summer of 1931
- **Foundational programme** from about Feb. 1932
- **System of natural deduction** finished by Sept. -32
- **Normalization** fall -32
- **Failure** of the original plan for a consistency proof *ca.* Jan. -33

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- What to do in an emergency? A thesis on pure logic
- The *Untersuchungen* was prepared in less than half a year, probably from March to end of May 1933

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- Work in logic from 1922 on, sustained by Bernays

Hertz systems

The rules of Hertz systems:

$a_1, \ldots, a_n \rightarrow a_i \quad$ for $1 \leqslant i \leqslant n$,

$$\frac{\left\{\begin{array}{l} a_{1_1}, \ldots, a_{1_n} \rightarrow b_1 \\ \vdots \\ a_{m_1}, \ldots, a_{m_k} \rightarrow b_m \end{array}\right. \qquad b_1, \ldots, b_m, c_1, \ldots, c_l \rightarrow c}{a_{1_1}, \ldots, a_{1_n}, \ldots, a_{m_1}, \ldots, a_{m_k}, c_1, \ldots, c_l \rightarrow c}$$

- The former are “immediate inferences,” the latter “syllogisms”

The rule of cut

Gentzen worked on Hertz systems in the summer of 1931 and noticed that "syllogisms" can be replaced by:

Gentzen's rule of cut in Hertz systems

$$\frac{a_1, \ldots, a_m, \rightarrow b \quad b, c_1, \ldots, c_n \rightarrow c}{a_1, \ldots, a_m, c_1, \ldots, c_n \rightarrow c}$$

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- the rule surfaced again in 1933

- reason was that Gentzen could not prove normalization for classical natural deduction

Linear derivations

- Linear derivations in Aristotle, Frege, Peano, Russell, Hilbert, Bernays, etc

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- Require a notation for referring to earlier conclusions

- No way to see clearly what depends on what

- Therefore composition of two derivations problematic

- Permutation of order of application of rules awkward

Trees

Frege had two-dimensional formulas and one-dimensional derivations when he should have had exactly the opposite!

Roy Dyckhoff, February 2009

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- two-dimensional inference figures from Hertz (1923)
- derivation trees a matter of course in Gentzen (1932)
- our habits derive from Gentzen but we are hardly aware of it

Natural deduction, September 1932

(from a stenographic manuscript, courtesy Christian Thiel)

23.IX.32 *Inference schemes:*

AI $\frac{A \quad B}{A\&B}$

AE $\frac{A\&B}{A}$ $\frac{A\&B}{B}$

OI $\frac{A}{A \vee B}$ $\frac{B}{A \vee B}$

OE

$$\frac{A \vee B \quad \begin{matrix} A \\ C \end{matrix} \quad \begin{matrix} B \\ C \end{matrix}}{C}$$

FI

$$\frac{\begin{matrix} A \\ \vdots \\ B \end{matrix}}{A \rightarrow B}$$

FE $\frac{A \quad A \rightarrow B}{B}$

Five forms of natural calculi, September 1932

UI

$$\frac{Pa}{x\ Px}$$

UE

$$\frac{x\ Px}{Ph}$$

EI

$$\frac{Ph}{Ex\ Px}$$

EE

$$\frac{Ex\ Px}{Pa}$$

RA

$$\frac{B \quad \begin{matrix} A \\ \vdots \\ \neg B \end{matrix}}{\neg A}$$

REND

$$\frac{\neg\neg A}{A}$$

CI

$$\frac{P1 \quad \begin{matrix} Pa \\ \vdots \\ Pa' \end{matrix}}{Ph}$$

Natural deduction, September 1932

NEE	*W*	*D*	*RA2*	*H(eyting)*
$\dfrac{\begin{matrix} A \\ \vdots \\ B \end{matrix} \quad \neg B}{\neg A}$	$\neg . A \& \neg A$	$A \vee \neg A$	$\dfrac{\begin{matrix} \neg A \\ \vdots \\ B \end{matrix} \quad \neg B}{A}$	$\dfrac{A \quad \neg A}{B}$

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- take the above as a tentative list of possible rules

- notation not yet absolutely clear

Five forms of natural calculi, September 1932

Formal shape of proof: Each proposition conclusion of at most one inference, no circle in inference.

T *Tree form: Each proposition except for the endproposition a premiss in exactly one inference.*

N *Net form: Each proposition except for the endproposition premiss in at least one inference, possibly several.*

Variable conditions. . .

Condition 10: There is a linear order of propositions in a proof, in which each conclusion comes after its premisses, and in which no common free variable appears before its EE.

Condition 30: In a UI, there must not occur any common variable that depends on the all-variables.

Natural deduction, September 1932

Division of the inference schemes:

(C) AI to EE.

Intuitionistic (I): AI to RA, H.

Classical (K): AI to REND.

Natural deduction, September 1932

Division of the inference schemes:

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- this goes exactly as it should, especially, *H* not needed in *K*

Important forms of proof:

NNA: natural net proof general form, i.e., net form with conditions 6.2; 11, 10, 30, 33. (sheet 78.3–79.)

N1: Tree form, with conditions 6.2; 27, Bb4. (Bb 1, Bb3, Bb 4.) (sheet 66.3.)

N2: Tree form, inference forms AI,AE,UI,UE,RA and REND, with condition 6.2; Bb 4 (only the special case of condition 32.) (sheet 80.2)

NN2: Net form, inference forms AI,AE,UI,UE,RA and REND, with condition 6.2; 11. (sheet 84.1)

NO: Tree form, with condition 6.2; 10, 30, 33. Corresponds to NNA, condition 11 and superfluous because of the tree form. For complete "generality" there is still missing the admission of several EE's with the same variable. . .

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- basic insight about "peaks" by September 23

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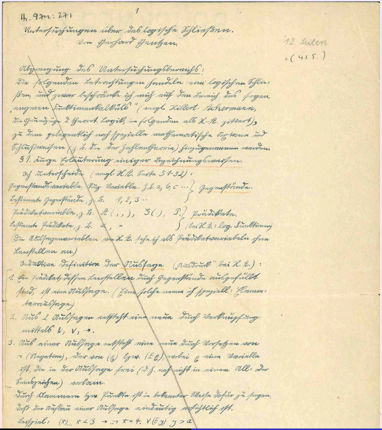
- normalization first conjectured in the handwritten thesis ms:

The early thesis manuscript


Untersuchungen über das logische Schließen.
von Gerhard Gentzen.
12 Seiten
Abgrenzung des Untersuchungsbereichs:

Conjectured theorem:
If a logical proposition is provable in the calculus N1I, there is a proof for it in which only subpropositions of it appear, possibly with different variables.
The proof for this is not yet finished except for a partial calculus N2I that maintains from N1I only the inferences AI,AE,UI,UE,R, but gains importance by the following fact:
The question of the provability of a logical proposition in the classical "lower functional calculus" can be reduced the question of the provability of (another) logical proposition in the calculus N1I.

- "Conjectured" cancelled, detailed 13-page proof added as a separate chapter III (now in *BSL*, June 2008)

Plan for the thesis

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Chapter V: Extend the subformula property of natural deduction to the system of intuitionistic arithmetic, to prove the consistency of the latter

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- Anyone who does that fails

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- Without further ado, Gentzen just put the normalization proof aside and never mentioned it in print

Translations

- equivalence of ND and axiomatic logic in planned chapter II

- first derive the Hilbert-Ackermann 1928 axioms in ND:

Equivalence with the H.A.-formalism.

I. *A proof in HA can be reproduced in N2.*
I define:

$A \vee B$ *as* $\neg.\neg A \& \neg B$, $(Ex)Px$ *as* $\neg(x)\neg Px$,
$A \rightarrow B$ *as* $\neg.A\&\neg B$

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- the axioms and rules of *HA* are now reproduced in *N2*

From ND to AX

II. *A proof in N2*

Can be reproduced in HA. It is first transformed as follows:

One finds for each proposition A of the N2-proof all those assumptions above it the respective RAA's (CI's) of which still stand under A. These are $B_1, \dots B_\varphi$*. Then one substitutes A by* $B_1 \& \dots \& B_\varphi \rightarrow A$*.*

If A itself is an assumption, $A \rightarrow A$ *takes its place.*

All initial propositions are by now correct in HA. For mathematical axioms have remained thus, and assumptions have become $A \rightarrow A$*.*

From ND to AX

The inferences read now as:

$$\text{AI}\quad \frac{D \rightarrow A \quad E \rightarrow B}{D\&E \rightarrow A\&B} \qquad \text{AE}\quad \frac{C \rightarrow A\&B}{C \rightarrow \begin{matrix} A \\ B \end{matrix}} \qquad \text{UE}\quad \frac{C \rightarrow (x)Px}{C \rightarrow Ph} \qquad \text{TND}\quad \frac{C \rightarrow \neg\neg A}{C \rightarrow A}$$

These are correct HA-inferences. Some more attention is required by UI, RAA, CI:

$$\text{UI}\quad \frac{C \rightarrow Pa}{C \rightarrow (x)Px}$$

This is a regular all-scheme, because by condition 2, a does not occur in $(x)Px$, and because of condition 3, neither in C.

I show how the second step of the tr. goes (just for AI):

$$\dfrac{\dfrac{(D \rightarrow A) \rightarrow ((E \rightarrow B) \rightarrow (D\&E \rightarrow A\&B)) \quad D \rightarrow A}{(E \rightarrow B) \rightarrow (D\&E \rightarrow A\&B)} \quad E \rightarrow B}{D\&E \rightarrow A\&B}$$

NL3

- a later addition to the thesis manuscript with the title

Calculus NL3, "natural-logistic" parallel calculus.

I explain the concept: "Proof of a proposition B from the propositions $A_1, \ldots A_r$*."*

I shall use the following symbolic writing when one such is at hand: $A_1, \ldots A_r \supset B$ *to be read as: B is provable from* $A_1, \ldots A_r$*. This is a "sentence" in the sense of P. Hertz.*

NL3

Now the recursive explanation of "proof":

I) *A is a proof of A from A.*

"natural" writing of this "proof": *symb. wr. for its formation:*

A $\quad$ $A \supset A$

(tautological sentence)

1) $\dfrac{(x)Px}{Ph}$ *is a proof of Ph from* $(x)Px$.

natural writing: *symb. wr. for the form. of the proof:*

UE) $\dfrac{(x)Px}{Ph}$ $\quad$ $(x)Px \supset Ph$

NL3

The following proofs are formed analogously:

EI) $\dfrac{Ph}{(Ex)Px}$ $\qquad Ph \supset (Ex)Px$

AE) $\dfrac{A\&B}{A} \quad \dfrac{A\&B}{B}$ $\qquad A\&B \supset A \quad A\&B \supset B$

OI) $\dfrac{A}{A \vee B} \quad \dfrac{B}{A \vee B}$ $\qquad A \supset A \vee B \quad B \supset A \vee B$

FE) $\dfrac{A \quad A \rightarrow B}{B}$ $\qquad A, A \rightarrow B \supset B$

NL3

2) A proof of Pa from $A_1 \ldots A_\nu$ *gives (through the joining of* $(x)Px$*, to below) rise to a proof of* $(x)Px$ *from* $A_1 \ldots A_\nu$*. Here A must not occur in* $A_1 \ldots A_\nu, (x)Px$.

natural writing: | *symbolic writing:* (*let* $\Gamma = A_1 \ldots A_\nu$)

$$\frac{A_1 \ldots A_\nu}{A_1 \ldots A_\nu}$$

$$\vdots$$

$$UI) \quad \frac{Pa}{(x)Px} \qquad\qquad \frac{\Gamma \supset Pa}{\Gamma \supset (x)Px}$$

a must not occur in $A_1 \ldots A_\nu, (x)Px$

- some more examples:

$$AI)\quad \dfrac{\dfrac{\dfrac{\Gamma}{\Gamma}}{\vdots\atop A}\qquad \dfrac{\dfrac{\Delta}{\Delta}}{\vdots\atop B}}{A\&B} \qquad\qquad \dfrac{\Gamma \supset A \quad \Delta \supset B}{\Gamma\ \Delta \supset A\&B}$$

$$FI)\quad \dfrac{\dfrac{\dfrac{\Gamma}{[A]\Gamma}}{\vdots\atop B}}{A \rightarrow B} \qquad\qquad \dfrac{[A]\Gamma \supset C}{\Gamma \supset A \rightarrow B}$$

Structural rules

II) *Joining of a condition* *(Thinning)*

from $\begin{array}{c}\Gamma\\ \vdots\\ B\end{array}$ *becomes* $\begin{array}{c}\Gamma\ A\\ \vdots\\ B\end{array}$

$$\frac{\Gamma \supset B}{\Gamma\ A \supset B}$$

Hanging one beside the other *(Cut)*

from $\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}$ *and* $\begin{array}{c}\Delta\ A\\ \vdots\\ B\end{array}$ *becomes* $\begin{array}{c}\quad\ \ \Gamma\\ \quad\ \ \vdots\\ \Delta\quad A\\ \vdots\\ B\end{array}$

$$\frac{\Gamma \supset A \qquad \Delta\ A \supset B}{\Gamma\ \Delta \supset B}$$

There is to each logical sign an axiom and an inference scheme.
All logical-sign-axioms can be replaced by inference schemes, namely, for example:

$$\frac{\Gamma \supset (x)Px}{\Gamma \supset Ph} \qquad \frac{\Gamma\ (Ex)Px \supset C}{\Gamma\ Ph \supset C}$$

$$\frac{\Gamma \supset A\&B}{\begin{array}{c}\Gamma \supset A\\ B\end{array}} \qquad \frac{A \vee B\ \Gamma \supset C}{\begin{array}{l}A\ \Gamma \supset C\\ B\end{array}} \qquad \frac{\Gamma \supset A \rightarrow B}{A\ \Gamma \supset B}$$

- there are no left rules, so the calculus is still "natural deduction in SC style," the calculus of Gentzen (1936) denoted *NLI* (resp. *NLK*) in some manuscripts

A classical calculus

Calculus *LDK*. (Logistic-dualistic.) (classical.)

A **sentence** *is written:*

$$A_1, \ldots, A_\mu \supset B_1, \ldots B_\nu \qquad \textit{in short:}\ \Gamma \supset \Delta$$

Structure-*axiom:* $A \supset A$

Structure-inferences:

Thinning: $\dfrac{\Gamma \supset \Delta}{\Theta\ \Gamma \supset \Delta\ \Lambda}$ *Cut:* $\dfrac{\Gamma \supset \Delta\ A \quad A\ \Theta \supset \Lambda}{\Gamma\ \Theta \supset \Delta\ \Lambda}$

Logical sign-axioms and inferences:
1. The propositional calculus.
Axioms:

&	$\vee$
$A\&B \supset A$	$A \supset A \vee B$
$A\&B \supset B$	$B \supset A \vee B$
$A, B \supset A\&B$	$A \vee B \supset A, B$

$\neg$

$A, \neg A \supset$	$\supset A, \neg A$
(Law of contradiction)	*(Law of excluded middle)*

Remarks: $\supset A$ *means intuitively: A is correct,*

NDK

2. The predicate calculus. *Axioms:*

$$(\ ) \qquad\qquad (E)$$

$$(x)Px \supset Ph \qquad\qquad Ph \supset (Ex)Px$$

Inferences:

$$(\ ) \qquad\qquad (E)$$

$$\frac{\Gamma \supset Pa}{\Gamma \supset (x)Px} \qquad\qquad \frac{Pa \supset \Gamma}{(Ex)Px \supset \Gamma}$$

a must not occur in the conclusion.

- now we have, for the first time, a **left rule**

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- only remaining copy of original manuscript “vernichtet" last year! (Eckart Menzler-Trott, e-mail 13 Nov.)

- Hidden in Gentzen's thesis is a sequent calculus *LIG* with half rules, half **groundsequents**.

- It is in the last section that translates:

AX to *NI* to *LI* (to *LIG*) to *AX*

$$A\&B \rightarrow A, \quad A\&B \rightarrow B, \qquad \frac{\Gamma \longrightarrow A \quad \Gamma \longrightarrow B}{\Gamma \longrightarrow A\&B}R\&$$

$$\frac{A, \Gamma \longrightarrow C \quad B, \Gamma \longrightarrow C}{A \vee B, \Gamma \longrightarrow C}L\vee \qquad A \rightarrow A \vee B, \quad B \rightarrow A \vee B$$

$$\frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B}R\supset \qquad A \supset B, A \rightarrow B$$

$$A, \neg A \rightarrow$$

$$\forall x A(x) \rightarrow A(t) \qquad \frac{\Gamma \longrightarrow A(y)}{\Gamma \longrightarrow \forall x A(x)}R\forall$$

$$\frac{A(y), \Gamma \longrightarrow C}{\exists x A(x), \Gamma \longrightarrow C}L\exists \qquad A(t) \rightarrow \exists x A(x)$$

LI to *LIG*:

$$\frac{A, \Gamma \rightarrow \Theta}{A\&B, \Gamma \rightarrow \Theta}\,L\& \quad \rightsquigarrow \quad \frac{\overset{Gs1}{A\&B \rightarrow A} \quad A, \Gamma \rightarrow \Theta}{A\&B, \Gamma \rightarrow \Theta}\,Cut$$

$$\frac{\Gamma \rightarrow A}{\neg A, \Gamma \rightarrow}\,L\neg \quad \rightsquigarrow \quad \cfrac{\cfrac{\Gamma \rightarrow A \quad \cfrac{\overset{Gs7}{\neg A, A \rightarrow}}{A, \neg A \rightarrow}\,E}{\Gamma, \neg A \rightarrow}\,Cut}{\neg A, \Gamma \rightarrow}\,E^*$$

Some results:

- Translation gives a complete class of *LIG*-derivations with cuts *but* with the subformula property.

- Any *LIG*-derivation can be so transformed that it has the subformula property

ND and SC

Problem. Correspondence between derivations in ND and SC

Gentzen translates normal derivations into ones with cuts

ND and SC

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Gentzen translates normal derivations into ones with cuts

- two solutions:

A. change rules $\&E, \supset E, \forall E$ of ND to have isomorphism with cut-free derivations in Gentzen's original SC

B. restrict rules $L\&, L\supset, L\forall$ suitably to have isomorphism with normal derivations in Gentzen's original ND

A. General elimination rules

- Write all E-rules in the style of $\vee E, \exists E$:

$$\frac{A\&B \qquad \begin{array}{c}\overset{1}{A}, \overset{1}{B}\\ \vdots \\ C\end{array}}{C}\,\&E,1 \qquad \frac{A \supset B \quad A \quad \begin{array}{c}\overset{1}{B}\\ \vdots \\ C\end{array}}{C}\,\supset E,1 \qquad \frac{\forall x A \qquad \begin{array}{c}\overset{1}{A(t/x)}\\ \vdots \\ C\end{array}}{C}\,\forall E,1$$

A. From ND to SC

- we define a root-first translation (JvP, *AML 2001*):

$$\dfrac{\dfrac{\begin{matrix}\Gamma & \Delta\\ \vdots & \vdots \\ A & B\end{matrix}}{A\&B}{\scriptstyle \&I} \qquad \begin{matrix}\overset{1}{A},\overset{1}{B},\Theta\\ \vdots\\ C\end{matrix}}{C}{\scriptstyle \&E,1} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\Gamma & \Delta\\ \vdots & \vdots \\ A & B\end{matrix}}{\Gamma,\Delta \to A\&B}{\scriptstyle R\&} \qquad \dfrac{\begin{matrix}A,B,\Theta\\ \vdots\\ C\end{matrix}}{A\&B,\Theta \to C}{\scriptstyle L\&}}{\Gamma,\Delta,\Theta \to C}{\scriptstyle Cut}$$

- result: **derived** *major prem's of E-rules give cuts*

Normal case

Normality $\equiv_{df}$ major prem's of E-rules **assumptions:**

$$\dfrac{A\&B \qquad \begin{matrix}\overset{1}{A}, \overset{1}{B}, \Theta \\ \vdots \\ C\end{matrix}}{C}\,\&E,1 \qquad \rightsquigarrow \qquad \dfrac{A\&B \to A\&B \qquad \dfrac{\begin{matrix}A, B, \Theta \\ \vdots \\ C\end{matrix}}{A\&B, \Theta \to C}\,L\&}{A\&B, \Theta \to C}\,Cut$$

- conclusion of *Cut* identical to its right premiss, so delete it
- result: *order of logical rules the same in ND and SC*

B. Change some sequent rules

- To do this in a transparent way, take Gentzen's **NI** rules as special cases of the general rules:

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$$\frac{A\&B \quad \overset{1}{A}}{A}\,\&E,1 \qquad \frac{A\&B \quad \overset{1}{B}}{B}\,\&E,1 \qquad \frac{A \supset B \quad A \quad \overset{1}{B}}{B}\,\supset E,1 \qquad \frac{\forall x A \quad \overset{1}{A(t/x)}}{A(t/x)}\,\forall E,1$$

- now apply isomorphic translation to Gentzen's rules:

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\&B\end{matrix} \quad \overset{1}{A}}{A}\,\&E{,}1 \qquad \rightsquigarrow \qquad \dfrac{\begin{matrix}\vdots\\ \Gamma \rightarrow A\&B\end{matrix} \quad \dfrac{A \rightarrow A}{A\&B \rightarrow A}\,L\&}{\Gamma \rightarrow A}\,Cut$$

- now apply isomorphic translation to Gentzen's rules:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\&B\end{array} \quad \stackrel{1}{A}}{A}\,{\scriptstyle \&E,1} \quad \rightsquigarrow \quad \dfrac{\Gamma \stackrel{\vdots}{\longrightarrow} A\&B \quad \dfrac{A \rightarrow A}{A\&B \rightarrow A}\,{\scriptstyle L\&}}{\Gamma \rightarrow A}\,{\scriptstyle Cut}$$

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A \supset B\end{array} \quad \begin{array}{c}\Delta\\ \vdots\\ A\end{array} \quad \stackrel{1}{B}}{B}\,{\scriptstyle \supset E,1} \quad \rightsquigarrow \quad \dfrac{\Gamma \stackrel{\vdots}{\longrightarrow} A \supset B \quad \dfrac{\Delta \stackrel{\vdots}{\longrightarrow} A \quad B \rightarrow B}{A \supset B, \Delta \rightarrow B}\,{\scriptstyle L\supset}}{\Gamma, \Delta \rightarrow B}\,{\scriptstyle Cut}$$

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$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A\&B\end{matrix} \quad \overset{1}{A}}{A}\,\&E,1 \quad \rightsquigarrow \quad \dfrac{\Gamma \overset{\vdots}{\rightarrow} A\&B \quad \dfrac{A \rightarrow A}{A\&B \rightarrow A}\,L\&}{\Gamma \rightarrow A}\,Cut$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ A \supset B\end{matrix} \quad \begin{matrix}\Delta\\ \vdots\\ A\end{matrix} \quad \overset{1}{B}}{B}\,\supset E,1 \quad \rightsquigarrow \quad \dfrac{\Gamma \overset{\vdots}{\rightarrow} A \supset B \quad \dfrac{\Delta \overset{\vdots}{\rightarrow} A \quad B \rightarrow B}{A \supset B, \Delta \rightarrow B}\,L\supset}{\Gamma, \Delta \rightarrow B}\,Cut$$

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ \forall x A\end{matrix} \quad \overset{1}{A(t/x)}}{A(t/x)}\,\forall E,1 \quad \rightsquigarrow \quad \dfrac{\Gamma \overset{\vdots}{\rightarrow} \forall x A \quad \dfrac{A(t/x) \rightarrow A(t/x)}{\forall x A \rightarrow A(t/x)}\,L\forall}{\Gamma \rightarrow A(t/x)}\,Cut$$

Result for solution B:

Result 1. *Derivations in NI are isomorphic to derivations in sequent calculus when the following restrictions apply:*

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(ii) *Premisses of rules L& and L$\forall$, and the second premiss of rule L$\supset$, are initial sequents and the principal formulas cut formulas in the next rule.*

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- These results were considered "too easy" to be publishable

ND and SC

Problem. Explore solutions A and B for a convincing correspondence between normalization and cut elimination

SC with groundsequents

Problem. Find a proof of cut elimination for *LIG, LKG* that does not use multicut

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- several other sequent calculi with similar open questions can be found in the Gentzen papers

Göttingen in 1933:

- Bernays was fired in April, so Weyl signed Gentzen's thesis evaluation on 9 June, analogously to the case of Noether and Ludwig Schwartz

- Weyl became director of the Göttingen mathematics institute, after Courant who got fired, and wrote in April 1935:

In the spring of 1933 the storm of the National Revolution broke over Germany. The Göttingen mathematical-natural scientific faculty, for building up and consolidation of which Klein and Hilbert had worked for decades, was struck at its roots. After an interregnum of one day by Neugebauer, I had to take over the direction of the Mathematical Institute. But Emmy Noether, as well as many others, was prohibited from participation in all academic activities, and finally her venia legendi, as well as her "Lehrauftrag" and the salary going with it, were withdrawn. A stormy time of struggle like this one we spent in Göttingen in the summer of 1933 draws people together, thus I have a particularly vivid recollection of these months.

At the time of writing the above, Weyl make strong attempts to bring Gentzen to Princeton:

Yesterday I received a new letter from Professor Weyl. He wrote that the Rockefeller Foundation has recently ceased entirely the giving out of stipends for mathematics, physics, and chemistry, and that also the Institute in Princeton could grant me no stipend this year. He hoped, however, to be able to obtain an invitation from the Institute next year.

(Gentzen, 27 February 1935)

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The representative of the Dozentenschaft in Göttingen could only find out about Gentzen that two scientific works of the applicant exist, which deal with "mathematical logic and similar things" and are deemed "unproductive" by the local specialists. In addition through a little snooping it was found that a few days earlier a letter for the applicant arrived from Jerusalem: "which anyhow allows one to conclude good connections to the representatives of the chosen people."

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- indeed, Gentzen kept up his correspondence with Bernays until the war, and he corresponded also with Fraenkel in 1935, risking the scholarship and a nomination to Hilbert's assistant

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- the other one was:

Elektronenbahnen in axialsymmetrischen Feldern mit Anwendung auf kosmische Probleme

- written for another “non-Aryan," Lothar Nordheim

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- as to "Heil Hitler," it was an obligation of civil servants, with detailed instructions about the salute lest one gets fired

The statement by the teachers' union shows that the Göttingen Nazis knew of Gentzen's two publications. They also knew that he corresponded with expelled Jews, but that was it.

I see no sympathies in either direction