The Interface between P and NP

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3-SAT

- Where are the hard 3-SAT problems?
- Sample randomly generated 3-SAT
  - Fix number of clauses, \( l \)
  - Number of variables, \( n \)
  - By definition, each clause has 3 variables
  - Generate all possible clauses with uniform probability
Random 3-SAT

Which are the hard instances?
- around $l/n = 4.3$

What happens with larger problems?
Why are some dots red and others blue?
This is a so-called “phase transition”
Where did this all start?

- At least as far back as 60s with Erdos & Renyi
  - thresholds in random graphs
- Late 80s
  - pioneering work by Karp, Purdom, Kirkpatrick, Huberman, Hogg ...
- Flood gates burst
  - Cheeseman, Kanefsky & Taylor’s IJCAI-91 paper
What do we know about this phase transition?

- It’s shape
  - Step function in limit [Friedgut 98]

- It’s location
  - Theory puts it in interval:
    \[ 3.42 < l/n < 4.506 \]
  - Experiment puts it at:
    \[ l/n = 4.2 \]
3SAT phase transition

- Lower bounds (hard)
  - Analyse algorithm that almost always solves problem
  - Backtracking hard to reason about so typically without backtracking
    - Complex branching heuristics needed to ensure success
    - But these are complex to reason about
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method
    
    For any statistic $X$
    $$\text{prob}(X\geq 1) \leq \mathbb{E}[X]$$
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

For any statistic $X$

$$\text{prob}(X \geq 1) \leq E[X]$$

No assumptions about the distribution of $X$ except non-negative!
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method
  
  For any statistic $X$
  
  $\Pr(X \geq 1) \leq E[X]$
  
  Let $X$ be the number of satisfying assignments for a 3SAT problem
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

For any statistic $X$

$\text{prob}(X=1) \leq E[X]$  

Let $X$ be the number of satisfying assignments for a 3SAT problem

*The expected value of $X$ can be easily calculated*
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

For any statistic X
\[ \text{prob}(X \geq 1) \leq E[X] \]

Let X be the number of satisfying assignments for a 3SAT problem
\[ E[X] = 2^n \times (7/8)^l \]
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

  For any statistic \( X \)
  
  \[
  \text{prob}(X \geq 1) \leq E[X]
  \]

  Let \( X \) be the number of satisfying assignments for a 3SAT problem
  
  \[
  E[X] = 2^n \times (7/8)^l
  \]

  If \( E[X] < 1 \), then \( \text{prob}(X \geq 1) = \text{prob}(\text{SAT}) < 1 \)
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

For any statistic $X$

$$\text{prob}(X \geq 1) \leq \mathbb{E}[X]$$

Let $X$ be the number of satisfying assignments for a 3SAT problem

$$\mathbb{E}[X] = 2^n \times (7/8)^l$$

If $\mathbb{E}[X] < 1$, then $2^n \times (7/8)^l < 1$
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

For any statistic X
\[
\text{prob}(X \geq 1) \leq E[X]
\]
Let X be the number of satisfying assignments for a 3SAT problem
\[
E[X] = 2^n \times (7/8)^l
\]
If \(E[X] < 1\), then
\[
2^n \times (7/8)^l < 1
\]
\[
n + l \log_2(7/8) < 0
\]
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - E.g. Markov (or 1st moment) method

For any statistic $X$

$$\Pr(X \geq 1) \leq E[X]$$

Let $X$ be the number of satisfying assignments for a 3SAT problem

$$E[X] = 2^n \cdot (7/8)^l$$

If $E[X] < 1$, then

$$2^n \cdot (7/8)^l < 1$$

$$n + l \cdot \log_2(7/8) < 0$$

$$l/n > 1/\log_2(8/7) = 5.19...$$
3SAT phase transition

- Upper bounds (easier)
  - Typically by estimating count of solutions
  - To get tighter bounds than 5.19, can refine the counting argument
    - E.g. not count all solutions but just those maximal under some ordering
Random 2-SAT

- 2-SAT is P
  - linear time algorithm

- Random 2-SAT displays "classic" phase transition
  - $l/n < 1$, almost surely SAT
  - $l/n > 1$, almost surely UNSAT
  - complexity peaks around $l/n=1$

$x_1 \lor x_2$, $-x_2 \lor x_3$, $-x_1 \lor x_3$, ...

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Phase transitions in P

- 2-SAT
  - $\forall n = 1$
- Horn SAT
  - transition not “sharp”
- Arc-consistency
  - rapid transition in whether problem can be made AC
  - peak in (median) checks
Phase transitions above NP

- PSpace
  - QSAT (SAT of QBF)
    \[ \forall x_1 \exists x_2 \forall x_3 . x_1 \lor x_2 \land \neg x_1 \lor x_3 \]
Phase transitions above NP

- **PSpace-complete**
  - QSAT (SAT of QBF)
  - stochastic SAT
  - modal SAT

- **PP-complete**
  - polynomial-time probabilistic Turing machines
  - counting problems
  - \#SAT(\geq 2^n/2)
Exact phase boundaries in NP

Random 3-SAT is only known within bounds
- $3.42 < \frac{l}{n} < 4.506$

Recent result gives an exact NP phase boundary
- 1-in-$k$ SAT at $\frac{l}{n} = \frac{2}{k(k-1)}$

Are there any NP phase boundaries known exactly?
Backbone

- Variables which take fixed values in all solutions
  - alias unit prime implicates

- Let $f_k$ be fraction of variables in backbone
  - in random 3-SAT
    - $l/n < 4.3$, $f_k$ vanishing (otherwise adding clause could make problem unsat)
    - $l/n > 4.3$, $f_k > 0$

  discontinuity at phase boundary!
Backbone

- Search cost correlated with backbone size
  - if $f_k$ non-zero, then can easily assign variable “wrong” value
  - such mistakes costly if at top of search tree
- One source of “thrashing” behaviour
  - can tackle with randomization and rapid restarts

Can we adapt algorithms to offer more robust performance guarantees?
Backbone

- Backbones observed in structured problems
  - quasigroup completion problems (QCP)

- Backbones also observed in optimization and approximation problems
  - coloring, TSP, blocks world planning ...

Can we adapt algorithms to identify and exploit the backbone structure of a problem?
2+p-SAT

- Morph between 2-SAT and 3-SAT
  - fraction $p$ of 3-clauses
  - fraction $(1-p)$ of 2-clauses

- 2-SAT is polynomial (linear)
  - phase boundary at $ln = 1$
  - but no backbone discontinuity here!

- 2+p-SAT maps from P to NP
  - $p > 0$, 2+p-SAT is NP-complete
2+p-SAT phase transition
2+p-SAT phase transition

\[ \frac{l}{n} \] vs \( p \)
2+p-SAT phase transition

- Lower bound
  - are the 2-clauses (on their own) UNSAT?
  - n.b. 2-clauses are much more constraining than 3-clauses

- \( p \leq 0.4 \)
  - transition occurs at lower bound
  - 3-clauses are not contributing!
2+p-SAT backbone

- $f_k$ becomes discontinuous for $p>0.4$
  - but NP-complete for $p>0$!
- search cost shifts from linear to exponential at $p=0.4$
- similar behavior seen with local search algorithms

Search cost against $n$
2+p-SAT trajectories

- Input 3-SAT to a SAT solver like Davis Putnam
- REPEAT assign variable
  - Simplify all unit clauses
  - Leaving subproblem with a mixture of 2 and 3-clauses
- For a number of branching heuristics (e.g. random,..)
  - Assume subproblems sample uniformly from 2+p-SAT space
  - Can use to estimate runtimes!
2+p-SAT trajectories

![Graph showing 2+p-SAT trajectories with SAT and UNSAT regions]
Beyond 2+p-SAT

- Optimization
  - MAX-SAT

- Other decision problems
  - 2-COL to 3-COL
  - Horn-SAT to 3-SAT
  - XOR-SAT to 3-SAT
  - 1-in-2-SAT to 1-in-3-SAT
  - NAE-2-SAT to NAE-3-SAT
  - ..
Graph colouring

- Can we colour graph so that neighbouring nodes have different colours?

In $k$-COL, only allowed $k$ colours

- 3-COL is NP-complete
- 2-COL is P
Random COL

- Sample graphs uniformly
  - *n* nodes and *e* edges

- Observe colourability phase transition
  - random 3-COL is "sharp", \( e/n \approx 2.3 \)
  - **BUT** random 2-COL is not "sharp"

As \( n \to \infty \)

\[
\begin{align*}
\text{prob}(2\text{-COL} \ @ \ e/n=0) &= 1 \\
\text{prob}(2\text{-COL} \ @ \ e/n=0.45) &= \approx 0.5 \\
\text{prob}(2\text{-COL} \ @ \ e/n=1) &= 0
\end{align*}
\]
2+$p$-COL

- Morph from 2-COL to 3-COL
  - fraction $p$ of 3 colourable nodes
  - fraction $(1-p)$ of 2 colourable nodes

- Like 2+$p$-SAT
  - maps from P to NP
  - NP for any fixed $p>0$

- Unlike 2+$p$-SAT
  - maps from coarse to sharp transition
2+\rho-COL
$2+\rho$-COL sharpness

$p=0.8$
$2 + \rho$-COL search cost
2+\(p\)-COL

- Sharp transition for \(p>0.8\)
- Transition has coarse and sharp regions for \(0<p<0.8\)
- Problem hardness appears to increase from polynomial to exponential at \(p=0.8\)
- 2+\(p\)-COL behaves like 2-COL for \(p<0.8\)

- NB sharpness alone is not cause of complexity since 2-SAT has a sharp transition!
Location of phase boundary

- For sharp transitions, like $2+p$-SAT:
  
  As $n \to \infty$, if $\frac{l}{n} = c + \epsilon$, then UNSAT
  
  $\frac{l}{n} = c - \epsilon$, then SAT

- For transitions like $2+p$-COL that may be coarse, we identify the start and finish:
  
  $\delta_{2+p} = \sup \left\{ \frac{e}{n} \mid \text{prob}(2+p\text{-colourable}) = 1 \right\}$
  
  $\gamma_{2+p} = \inf \left\{ \frac{e}{n} \mid \text{prob}(2+p\text{-colourable}) = 0 \right\}$
Basic properties

- **monotonicity:** $\delta \leq \gamma$
- **sharp transition iff** $\delta = \gamma$
- **simple bounds:**
  \[
  \begin{align*}
  \delta_2 + p &= 0 \text{ for all } p < 1 \\
  \gamma_2 &\leq \gamma_2 + p \leq \min(\gamma_3, \gamma_2 / 1 - p)
  \end{align*}
  \]
2$+\rho$-COL phase boundary
XOR-SAT

- XOR-SAT
  - Replace or by xor
  - XOR $k$-SAT is in P for all $k$

- Phase transition
  - XOR 3-SAT has sharp transition
  - $0.8894 \leq l/n \leq 0.9278$ \cite{Creognou01}
  - Statistical mechanics gives $l/n = 0.918$ \cite{Franz01}
XOR-SAT to SAT

- Morph from XOR-SAT to SAT
  - Fraction \((1-p)\) of XOR clauses
  - Fraction \(p\) of OR clauses
- NP-complete for all \(p>0\)
  - Phase transition occurs at:
    - \(0.92 \leq l/n \leq \min(0.92/1-p, 4.3)\)
- Upper bound appears loose for all \(p>0\)
  - Polynomial subproblem does not dominate!
  - 3-SAT contributes (cf 2\(+p\)-SAT, 2\(+p\)-COL)
Other morphs between \( P \) and \( NP \)

- **NAE 2+p-SAT**
  - NAE = not all equal
  - NAE 2-SAT is P, NAE 3-SAT is NP-complete

- **1-in-2+p-SAT**
  - 1-in-\( k \) SAT = exactly one in \( k \) literals true
  - 1-in-2 SAT is P, 1-in-3 SAT is NP-complete

- …
NAE to SAT

- Morph between two NP-complete problems
  - Fraction \((1-p)\) of NAE 3-SAT clauses
  - Fraction \(p\) of 3-SAT clauses

- Each NAE 3-SAT clause is equivalent to two 3-SAT clauses
  - NAE 3-SAT phase transition occurs around \(l/n = 2.1\)
    - Tantalisingly close to half of 4.2
  - \(NAE(a,b,c) = or(a,b,c) \& or(-a,-b,-c)\)
    - Can we ignore many of the correlations that this encoding of NAE SAT into SAT introduces?
NAE to SAT

- Compute “effective” clause size
  - Consider $(1-p)l$ NAE 3-SAT clauses and $pl$ 3-SAT clauses
  - These behave like $2(1-p)l$ 3-SAT clauses and $pl$ 3-SAT clauses
  - That is, $(2-p)l$ 3-SAT clauses
  - Hence, effective clause to variable ratio is $(2-p)l/n$

- Plot prob(satisfiable) and search cost against $(2-p)l/n$
NAE to SAT

\[ \text{Prob(sat)} \]

\[ (2-p)l/n \]

Search cost

\[ (2-p)l/n \]
Conclusions

- There’s rich structure to be found between P and NP
- Problem classes like 2+p-SAT and 2+p-COL help us understand the onset of intractability
- NP-completeness isn’t everything!
  - Next lecture: the impact that structure has on problem hardness